The Mathematics Consortium

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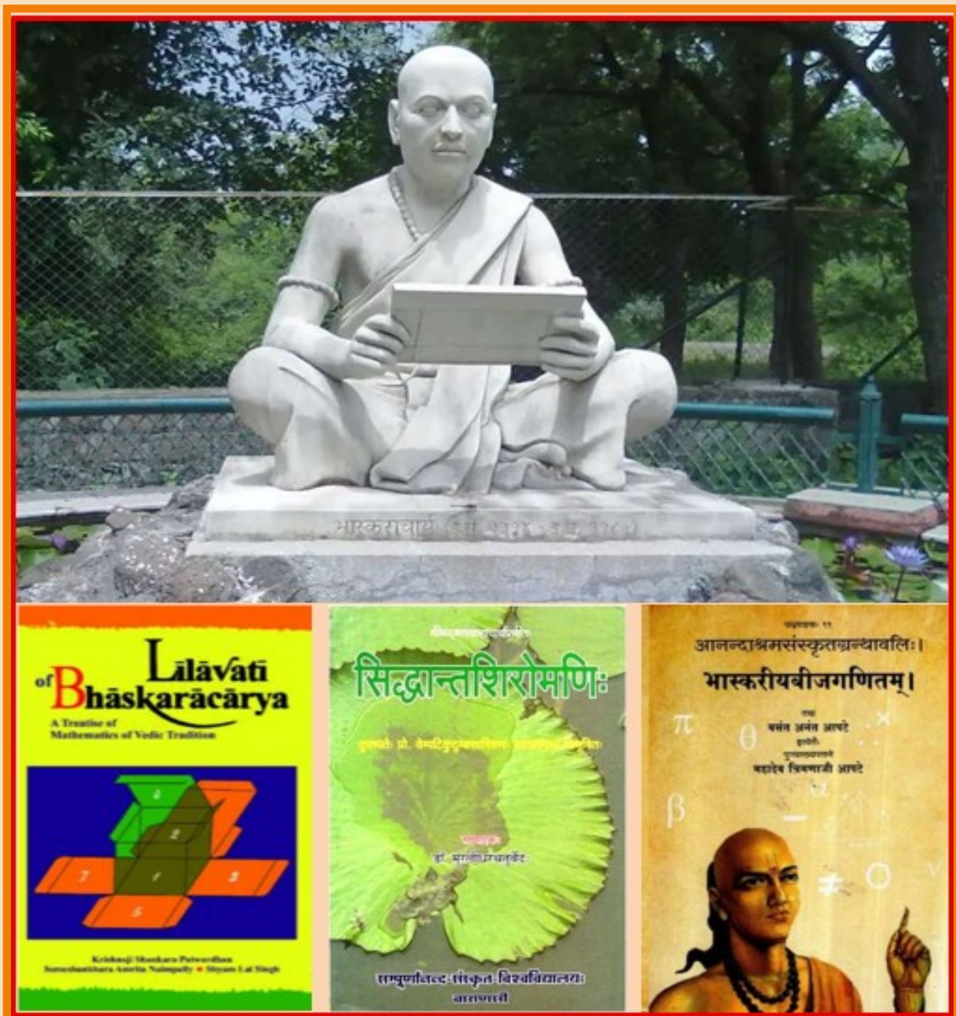
BHASKARACHARYA (1114-1185)



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Contents

***About the Cover Page**: The top image on the front cover is a statue of the celebrated 12^{th} century Indian mathematician Bhaskaracharya (also known as Bhaskara II) at Patna, a town about 18 km southwest of Chalisgaon (in the Jalgaon district of Maharashtra), which is of historic and touristic interest, on account of an inscription, located in the Patnadevi temple, commemorating a grant from a local ruler, in 1207, to support Changadeva, grandson of Bhaskaracharya, founding a school to pursue studies in Bhaskaracharya's work. The inscription features, in particular, the family tree of Bhaskaracharya. The other images are of front covers of the books Līlavatī, Siddhānta Śiromaṇī, and Bījagaṇitaṃ written by him.*

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From the Editors' Desk

In the ever-changing world around us it is imperative that the curricula at various levels of education be updated periodically. It is also natural that agencies such as the University Grants Commission (UGC) should take the lead in spearheading the revision, in respect of colleges and higher education institutions. In performing the task, it is necessary to take into account the emerging aspirations on the one hand, as well accumulated wisdom concerning the educational needs of the society, and the development of the academic framework itself, in the global context. The future of the young people being anchored on the education they get, it is important not to be driven by passions, however honourable they may be.

The recent draft of Learning Outcomes-based Curriculum Framework (LOCF) for a 4-year undergraduate program put out by UGC is a case in point. It is hard to believe that the Committee involved has given serious thought to the long-term effects of what is proposed in the draft. While no doubt there is a need for incorporating study of Indian Knowledge Systems, as has been highlighted in the New Education Policy (NEP), the way proposed in the present document for implementing it is seen to be highly inept. One of the overwhelming lacunae is that in their passion for the cause, the Committee has missed out on a sense of proportion and, in particular, the core mathematical content has got drastically cut down, with many core courses missing, and some rushed through without leaving enough time for a proper grasp of the material. Not surprisingly the draft has come under serious criticism from the academic community, with the realization that graduates from the program will be missing the basic grounding in mathematics and would become unfit in various respects.

A fresh committee is now understood to have been appointed by the UGC for a comprehensive evaluation of the draft. We hope that the committee will take into consideration the feedback received from various quarters, and would put things in order, producing a curriculum of truly international standards.

In our Bulletin, it has been our endeavour to spread awareness regarding Indian Knowledge Systems along with the latest developments in Mathematics, in the wider mathematics community.

In the present issue, in Article 2, Dr. D. C. Srivastava discusses two proofs, one algebraic and another geometric, of the Pythagoras theorem, as given by the twelfth century Indian mathematician Bhaskaracarya in his book Bijaganitam.

In Article 1, Prof. S. A. Katre introduces cyclic vector spaces over fields and explains how to get all such vector spaces over various fields.

In Article 3, Dr. D. V. Shah gives an account of significant developments in the mathematical world during recent past, such as new way of detecting primes using partition functions, resolution of the Kervaire invariant problem, and a counterexample given by a 17 year old student disproving the Mizohata-Takeuchi conjecture.

In Article 4, Prof. S. G. Dani gives brief review on two papers, one on Graphs in the 1680s and another on the first and most elementary construction of real numbers.

A journey from MMC to IMC of Undergraduate Math Competition is aptly described by Prof. V. M. Sholapurkar in Article 5. Prof. Surinder Pal Singh Kainth gives, in Article 6, a brief report on International Conference on Algebra and Number Theory organised to celebrate 100 years of Prof. R. P. Bambah, and in Article 7, Prof. M. S. Raghunathan pays tributes to Prof. P. P. Divakaran, who passed away recently.

In the Problem Corner, Dr. Udayan Prajapati presents a solution to the first problem posed in the April 2025 issue and two problems, one from Algebra and another from Number theory, are also posed for our readers. Dr. Ramesh Kasilingam gives a calendar of academic events, planned for January, 2026 to May, 2026, in Article 9.

We are happy to bring out this second issue of Volume 7 in October, 2025. We thank all the authors, all the editors, our designers Mrs. Prajakta Holkar and Dr. R. D. Holkar, and all those who have directly or indirectly helped us in bringing out this issue on time.

Chief Editor, TMC Bulletin

1. Cyclic Vector Spaces

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ABSTRACT

[1]In this article we discuss cyclic vector spaces over fields such as $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$ and finite fields. We prove that the number of cyclic vector spaces of a given length n is finite and is a power of 2 if the characteristic of the field is 0, or n is coprime to the (prime) characteristic. The cyclic vector spaces over a finite field are called cyclic codes and they have ample applications in error correction.

1.1 INTRODUCTION

Let F be a field and V be a subspace of F^n. We say that V is a cyclic subspace of F^n if $(a_1, a_2, ..., a_n) \in V$ implies that $(a_n, a_1, ..., a_{n-1}) \in V$. As $V \subset F^n$, we also say that V is a vector space of length n over F. The question is "What are all the cyclic subspaces of F^n?" Trivially $\{0\}$ and F^n are cyclic subspaces of F^n. Are there any other?

Let $W_1 = \{(a, a, \dots, a) | a \in F\}$. Then W_1 is a 1-dimensional subspace of F^n and it is a cyclic subspace. Another example of a cyclic subspace is an $(n-1)$-dimensional subspace given by the single equation $x_1 + x_2 + \dots + x_n = 0$. Thus

$$W_2 = \{(a_1, a_2, \dots, a_n) | a_i \in F, 1 \le i \le n, a_1 + a_2 + \dots + a_n = 0\}$$

is a cyclic subspace of F^n. Note that W_1 and W_2 coincide if F is a field of characteristic 2 and $n = 2$. A way to generate a cyclic subspace of F^n is to start with a few vectors in F^n and take the vectors obtained from them by the n cyclic changes (including identity) and take the subspace of F^n generated by all these vectors. This subspace is a cyclic subspace of F^n and it is called the cyclic subspace of F^n (cyclically) generated by the original vectors. Let us denote the cyclic subspace (cyclically) generated by $\{v_1, v_2, \dots, v_k\}$ by $<< v_1, \dots, v_k >>$. For example, $<< (1, 0, 0, \dots, 0) >>=$ the subspace of F^n generated by

$$(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots (0, 0, 0, \dots, 1)$$

is a cyclic subspace of F^n, but unfortunately this subspace is the whole space F^n, thus we do not get a new cyclic subspace. This example illustrates that it is somewhat difficult to get new cyclic subspaces by this method unless we are lucky to choose right initial vectors. We shall see that a cyclic subspace of length n is cyclically generated by just one (suitable) vector in F^n. To get cyclic subspaces we use the concept of ideals in a ring.

1.2 IDEALS AND QUOTIENT RINGS IN $F[x]$

Let $F[x]$ be the ring of polynomials in one variable x. If I is any ideal in $F[x]$, then it is known that I is a principal ideal, i.e. I is generated by a single polynomial, say $f(x)$. If $f(x)$ is the zero polynomial, then $I = (0)$ and if $f(x)$ is the constant polynomial 1, then $I = F[x]$, the whole polynomial ring. Apart from these cases, we can say that I is generated by a monic polynomial $f(x) \in F[x]$ of degree ≥ 1. We write $I = (f(x))$.

The set denoted by $\frac{F[x]}{I}$ is the set of cosets of I in $F[x]$. It is a commutative ring with unity under the natural addition and multiplication of cosets and is called the quotient ring of $F[x]$ by I. The ideals of this ring are of the form $\frac{J}{I}$ where J is an ideal of $F[x]$ containing I. Any such ideal J

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[1]Talk at Bhaskaracharya Pratishthana, Pune, on 22nd July 2025 as Professor Shreeram Abhyankar Memorial Lecture.

should be of the form $(g(x))$. Here, as $J \supset I = (f(x))$, we must have $f(x) \in J = (g(x))$, so that $f(x) = g(x)h(x)$ for some $h(x) \in F[x]$, i.e. $g(x)$ is a divisor of $f(x)$ in $F[x]$. From this it follows that the set of all ideals containing $I = (f(x))$ consists of the set of all $(g(x))$ where $g(x)$ is a divisor of $f(x)$ in $F[x]$. To get distinct ideals of $\frac{F[x]}{(f(x))}$ we take distinct monic divisors of $f(x)$. Note also that 1 is considered as a divisor of $f(x)$ here and it gives rise to the whole ideal $J = F[x]$. For $J = (f(x))$, $\frac{J}{(f(x))}$ is the (0) ideal of the ring $\frac{F[x]}{(f(x))}$.

It thus follows that in $\frac{F[x]}{(f(x))}$, the number of ideals is equal to the number of monic divisors of $f(x)$ in $F[x]$ (including 1). If moreover, the factorisation of $f(x)$ is given as $f(x) = f_1(x)^{e_1}f_2(x)^{e_2}\dots f_r(x)^{e_r}$, where $f_i(x), 1 \leq i \leq r$, are distinct monic irreducible polynomials in $F[x]$, then the number of all monic divisors of $f(x)$ including 1 is $(e_1+1)(e_2+1)\dots(e_r+1)$. Hence the number of ideals J of $F[x]$ containing I, so also the number of ideals of the quotient ring $\frac{F[x]}{(f(x))}$, is $(e_1+1)(e_2+1)\dots(e_r+1)$. We thus have

Theorem 1. *For any proper ideal $I = (f(x))$ of $F[x]$, the number of ideals of $\frac{F[x]}{I}$ is finite and is equal to the number of (monic) divisors of $f(x)$. Every ideal of $\frac{F[x]}{I}$ is of the form $\frac{J}{I}$ where $J = (g(x))$ for some monic divisor $g(x)$ of $f(x)$.*

Remark. Similar result holds for any PID. If R is a PID, then any proper ideal of R is of the form $I = (a)$ where a is a nonzero nonunit. Any ideal of R/I is of the form J/I where $J = (b)$ for some divisor b of a. If b and b' are associates, i.e. $b' = ub$ for some unit of R, then b, b' generate the same ideal of R. The number of ideals of $R/(a)$ is finite and is equal to the number of divisors of a (considered upto associates). If $a = \pi_1^{e_1}\dots\pi_r^{e_r}$ where $\pi_1,\dots,\pi_r$ are nonassociates in pairs, then the number of ideals of $R/(a)$ is again $(e_1+1)(e_2+1)\dots(e_r+1)$.

In case R is a Dedekind domain and I is a nonzero nonunit ideal of R, we can write I as $P_1^{e_1}\dots P_1^{e_r}$, where $P_i, 1 \leq i \leq r$, are distinct prime ideals and again the same formula works for the number of ideals of R/I. The ring of algebraic integers in an algebraic number field is a Dedekind Domain and this result holds there too.

1.3 The Ideals of $\frac{F[x]}{(x^n-1)}$ as F-vector spaces and their dimension

Take $g(x)$ as a monic divisor of x^n-1. Let $I = (x^n-1)$. Then

"$g(x)$ generates a cyclic vector space over F of length n".

i.e. the ideal $(g(x))/I$ of $F[x]/I$ gives rise to a cyclic vector space of length n over F. We will see this in the next section.

If $g(x)$ is $a_0+a_1x+a_2x^2+\dots+x^k$ for $0 \leq k \leq n$, then in the ideal J generated by $g(x)$ in the polynomial ring $F[x]$, the elements are of the type $g(x)q(x)$, where $q(x) \in F[x]$. If $g(x)h(x) = x^n-1$, then divide $q(x)$ by $h(x)$ to get the remainder $r(x)$, which is 0 or of degree $< \deg(h(x)) = n - \deg(g(x))$. Thus, $q(x) = h(x)l(x) + r(x)$, giving

$$g(x)q(x) = g(x)(h(x)l(x) + r(x)) = (x^n-1)l(x) + g(x)r(x).$$

Hence, the polynomials in $J = (g(x))$ considered modulo x^n-1 are linearly generated over the field F by $g(x)$ times $1, x, x^2, \dots, x^{n-k-1}$. The cosets of these $n-k$ polynomials w.r.t. $I = (x^n-1)$ are F-linearly independent as elements of J/I. Hence the F-vector space J/I is of dimension $n-k$ where $k = \deg(g(x))$. If $g(x) = x^n-1$, then the coset of $g(x)$ generates the (0) ideal of $F[x]/I$ and $J/I = (0)$. If $g(x) = 1$, the coset of $g(x)$ generates the unit ideal of $F[x]/I$ which is a vector space of dimension n over F. We have proved

Theorem 2. *If $g(x) \in F[x]$ is a factor of x^n-1, then the ideal $\frac{(g(x))}{(x^n-1)}$ of the ring $\frac{F[x]}{(x^n-1)}$ is an F-vector space of dimension $n-k$, where $k = deg(g(x))$. The cosets of the $n-k$ polynomials $g(x)\{1, x, \dots, x^{n-k-1}\}$ form an F-vector space basis of the ideal $\frac{(g(x))}{(x^n-1)}$.*

1.4 Cyclic vector spaces as $F[x]/(x^n - 1)$-modules

Let V be a cyclic vector space over F of length n. Let $v = (x_0, x_1, \dots, x_{n-1})$. Define for $v \in V$ a multiplication by x as $x \cdot v = (x_{n-1}, x_0, x_1, \dots, x_{n-2})$. From this, using the fact that V is an F-vector space, we can define multiplication of v by a polynomial in $F[x]$. Then V becomes an $F[x]$-module. Since V is a cyclic vector space, we get $x^n \cdot v = v$, so $(x^n - 1) \cdot v = 0$. Hence the ideal $I = (x^n - 1)$ annihilates v, so V is an $F[x]/(x^n - 1)$-module, where $(x + I) \cdot v = x \cdot v$.
Define $\psi : V \to \frac{F[x]}{I}$ by $\psi(x_0, x_1, \dots, x_{n-1}) = x_0 + x_1 x + \dots + x_{n-1}x^{n-1} + I$. This map is an F-linear map. Also for $v \in V$,

$$\psi((x + I) \cdot v) = \psi(x \cdot v) = \psi((x_{n-1}, x_0, x_1, \dots, x_{n-2})) = x_{n-1} + x_0 x + x_1 x^2 + \dots + x_{n-2}x^{n-1} + I.$$

Now, $(x + I)\psi(v) = (x + I)(x_0 + x_1 x + \dots + x_{n-1}x^{n-1} + I) = x_0 x + x_1 x^2 + \dots + x_{n-1}x^n + I = x_{n-1} + x_0 x + x_1 x^2 + \dots + x_{n-2}x^{n-1} + I$. This shows that $\psi((x + I) \cdot v) = (x + I)\psi(v)$. Thus ψ is a homomorphism of $\frac{F[x]}{I}$-modules. Note that

$$F[x]/(x^n - 1) = \{a_0 + a_1 x + \dots + a_{n-1}x^{n-1} + (x^n - 1) | a_i \in F, 0 \le i \le n - 1\}$$

and F^n is in a natural bijective correspondence with it. The map ψ is a restriction of this natural map to V and ψ is a one-one map into $F[x]/(x^n - 1)$. V is thus isomorphic to an $F[x]/(x^n - 1)$-submodule of itself, i.e. an ideal of $F[x]/(x^n - 1)$.
Conversely if K is an ideal of $F[x]/(x^n - 1)$, and $m(x) + (x^n - 1)$ is in K, where $m(x) = a_0 + a_1 x + \dots + a_{n-1}x^{n-1}$, then $(x + (x^n - 1))(m(x) + (x^n - 1))$ is in K, so $xm(x) + (x^n - 1)$ is in K, so $x_0 x + x_1 x^2 + \dots + x_{n-1}x^n + (x^n - 1)$ is in K, so $x_{n-1} + x_0 x + x_1 x^2 + \dots + x_{n-2}x^{n-1} + (x^n - 1)$ is in K. This shows that the subspace of F^n, which is in the natural bijective correspondence with this ideal of $F[x]/(x^n - 1)$, is a cyclic subspace of F^n. We have thus proved

Theorem 3. *The cyclic subspaces of F^n are in natural bijective correspondence with the ideals of $F[x]/(x^n - 1)$. Every cyclic subspace of F^n comes from an ideal $(g(x))/(x^n - 1)$ of $F[x]/(x^n - 1)$, where $g(x)$ is a divisor of $x^n - 1$. The number of cyclic vector spaces of length n over F is thus the number of divisors of $x^n - 1$. If $x^n - 1 = g_1(x)^{e_1} \cdots g_r(x)^{e_r}$, where $g_i(x), 1 \le i \le r$ are distinct monic irreducible factors of $x^n - 1$ over F, then the number of cyclic spaces of length n over F is the number of monic divisors of $x^n - 1$, which is $(e_1 + 1) \dots (e_r + 1)$.*

Remark. If char.$F = 0$ or char.$F = p > 0$ and $(p, n) = 1$, then $x^n - 1$ and its derivative nx^{n-1} do not have a common factor over F, so $x^n - 1$ has no repeated factor. Thus $x^n - 1$ is a product of distinct monic irreducible factors over F. If the number of these irreducible factors is r, then the number of divisors of $x^n - 1$ and hence the number of cyclic vector spaces of length n over F is 2^r which is a power of 2. If char.$F = p > 0$ (p prime), and $p|n$, then write $n = p^k N$, where $(p, N) = 1$. If F is a finite field or more generally any perfect field of characteristic p, then write $x^n - 1 = x^{p^k N} - 1 = y^N - 1$, where $y = x^{p^k}$. Now $y^N - 1$ is a product of distinct irreducible polynomials in y, say r in number. Since F is a perfect field, every element in F has a p-th root in F, so each irreducible factor of $y^N - 1$ is a p^k-th power of an irreducible polynomial in x over F. Hence the number of divisors of $x^n - 1$ over F and so also the number of cyclic vector spaces of length n over F is $(p^k + 1)^r$, thus a power of $p^k + 1$.

1.5 Generation of cyclic vector spaces of length n over a field

Let F be a field and let $g(x)$ be a monic factor of $x^n - 1$ over F. (A nonmonic factor and the associated monic factor - obtained by dividing by the leading coefficient - will give rise to the same cyclic space.) Write $g(x) = a_0 + a_1 x + \dots + x^k, 0 \le k \le n$. Then $g(x)$ corresponds to the element $(a_0, a_1, \dots, 1, 0, \dots, 0)$ in F^n. Here the number of zeros at the end is $n - k - 1$. The polynomials $xg(x), \dots, x^{n-k-1}g(x)$ will correspond to the n-tuples obtained from the n-tuple $(a_0, a_1, \dots, 1, 0, \dots, 0)$

by the $n-k-1$ cyclic changes, until all the zeros at the end come in the beginning. We have thus got $n-k$ vectors in all, obtained from one vector corresponding to $g(x)$ by cyclic changes. These vectors are linearly independent over F. Since the dimension of the vector space is also $n-k$, these $n-k$ vectors will generate our cyclic space. It is not necessary to consider any further cyclic changes, and such vectors are going to be automatically in our cyclic space by suitable linear combinations of our $n-k$ vectors. By varying $g(x)$ we get all cyclic spaces of length n.
We thus see that a cyclic subspace of F^n is obtained from just one vector in F^n, which comes as above from the coefficients of a divisor $g(x)$ of x^n-1 in $F[x]$. This vector and its further $n-k-1$ cyclic translates generate over F our cyclic subspace. This method gives you all cyclic subspaces of F^n of length n over F.
Here we have written the polynomial $g(x)$ in the order of increasing powers of x. But that is not important if we want to find all cyclic subspaces. This is because if $g(x)$ is the minimal polynomial of some root α of x^n-1, then writing the coefficients of $g(x)$ in the opposite direction, we get a polynomial (called the reciprocal polynomial of $g(x)$) which is (an F-multiple of) the minimal polynomial of $1/\alpha$ which is also an n-th root of 1. So this polynomial is also a divisor of x^n-1. The product of reciprocal polynomials is equal to the reciprocal polynomial of the product. Hence the set of reciprocal polynomials of divisors of x^n-1 is the same as the set of divisors of x^n-1 (apart from possibly an F-multiple). Thus the order in which the coefficients of $g(x)$ are written is not important to get all cyclic spaces of length n, as long as we keep uniformity.

1.6 Examples of cyclic vector spaces

Let F be a field and $g(x)$ be a divisor of x^n-1 of degree k. Let V be the cyclic vector space of length n corresponding to $g(x)$.

1. If $k=0$, i.e. $g(x)=1$, then $\dim(V)=n$ and V is the full vector space F^n.
2. If $k=n$, i.e. $g(x)=x^n-1$, then $\dim(V)=0$ and V is the (0) vector space.
3. Take the factor $g(x)=x-1=-1+x$ of x^n-1. Let $n=6$. Using the coefficients -1 and 1, we see that the vector space V is generated by the 5 vectors $(-1,1,0,0,0,0), (0,-1,1,0,0,0), (0,0,-1,1,0,0), (0,0,0,-1,1,0), (0,0,0,0,-1,1)$. Elements in this vector space are of the type $(-a, a-b, b-c, c-d, d-e, e)$. We see here that the sum of all the entries is 0, i.e. the vectors in this space satisfy $x_1+x_2+x_3+x_4+x_5+x_6=0$. By dimension arguments (dimension = 5), the cyclic vector space obtained from the factor $x-1$ is equal to the vector space obtained from this linear equation. For a general n, the n-tuple $(-1,1,0,\ldots 0)$ of length n cyclically generates an $(n-1)$-dimensional cyclic subspace of length n given by the equation $x_1+\ldots+x_n=0$.
4. Consider the factor $1+x+\ldots+x^{n-1}$ of x^n-1. The cyclic vector space given by this factor is 1-dimensional and using the coefficients, it is generated by $(1,1,1,\ldots,1)$, and consists of vectors of the type $(a,a,a,\ldots,a)$.
5. To get other cyclic vector spaces use various divisors of x^n-1 over F.

1.7 Number of cyclic vector spaces over different fields

If char.$F=0$, or char.$F=p$ (prime) and $(n,p)=1$, then the number of cyclic vector spaces of length n is 2^m, where m is the number of (monic) distinct irreducible factors of x^n-1. Thus the maximum number of cyclic vector spaces in this case is 2^n.

1. For $F=\mathbb{C}$, we have $x^n-1=\prod_{k=0}^{n}(x-e^{2\pi ik/n})$ is a product of n distinct monic linear factors. Similarly for any algebraically closed field of characteristic 0, or of characteristic p where $(n,p)=1$, we get n distinct irreducible (linear) factors of x^n-1. Thus in this case $m=n$.

Note that this happens in any field F of char. 0 or with $(n, \text{char}.F) = 1$, which contains n (distinct) n-th roots of 1, e.g. the cyclotomic field $\mathbb{Q}(e^{2\pi i/n})$.

2. For $F = \mathbb{R}$, for n odd, the polynomial $x^n - 1$ has one real root $x = 1$, and other roots are in complex conjugate pairs. Each such pair gives rise to an irreducible quadratic factor of $x^n - 1$ over $\mathbb{R}$. So $m = 1 + (n-1)/2$. If n is even, $1, -1$ are the only real roots of $x^n - 1$ and the remaining roots are in conjugate pairs, so $m = 2 + (n-2)/2$.

3. Let $F = \mathbb{Q}$. We have $x^n - 1 = \prod_{k=0}^{n}(x - e^{2\pi ik/n})$. We wish to get irreducible factors over $\mathbb{Q}$. For this, note that this product is the product over all n-th roots of 1 in $\mathbb{C}$. Every n-th root of 1 is a primitive d-th root of 1 for some divisor d of n.

   We collect together the factors associated to primitive d-th roots of 1. Let $\Phi_d(x) = \prod_\zeta (x - \zeta)$, where ζ varies over primitive d-th roots of unity. Then $x^n - 1 = \prod_{d|n} \Phi_d(x)$. We have, $\Phi_1(x) = x - 1, \Phi_2(x) = x + 1$, and for an odd prime p, $\Phi_p(x) = x^{p-1} + x^{p-2} + \ldots + 1$. These are polynomials over $\mathbb{Q}$, moreover over $\mathbb{Z}$. We have $x^n - 1 = [\prod_{d|n, d<n} \Phi_d(x)]\Phi_n(x)$. By induction, we can prove that all $\Phi_n(x)$ are monic polynomials with integer coefficients. Thus $x^n - 1 = \prod_{d|n} \Phi_d(x)$ is a factorisation over integers. It is possible to show that each $\Phi_n(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$. The degree of $\Phi(x)$ is $\phi(n)$, where ϕ denotes the Euler function. These polynomials $\Phi_n(x)$ are called cyclotomic polynomials. Thus we get that $x^n - 1$ is a product of $d(n)$ irreducible (cyclotomic) polynomials over $\mathbb{Q}$, where $d(n)$ is the number of divisors of n. Thus for $F = \mathbb{Q}$, $m = d(n)$, and the number of cyclic spaces over $\mathbb{Q}$ is $2^{d(n)}$. To get the actual cyclic spaces we should get the irreducible polynomials $F_n(x)$. These can be obtained easily from the following formulae:

   $\Phi_{p^k}(x) = \Phi_p(x^{p^{k-1}}), p$ prime. If m is odd > 1, $\Phi_{2m} = \Phi_m(-x)$. If p is a prime, $p|m$, then $\Phi_{pm} = \Phi_m(x^p)$. If p is an odd prime, $p \nmid m$, then $\Phi_{pm} = \frac{\Phi_m(x^p)}{\Phi_m(x)}$. These formulae can be proved just by using the definition of $\Phi_n(x)$.

4. (i) Consider $F = \mathbb{F}_2$. If $n = 4$, consider $x^4 - 1 = (x-1)^4$, so there are 5 cyclic spaces over $\mathbb{F}_2$ of length 4 corresponding to the $5 = 4 + 1 = 2^2 + 1$ factors
   $1, x - 1 = x + 1, (x-1)^2 = x^2 + 1, (x-1)^3 = x^3 + x^2 + x + 1, (x-1)^4 = x^4 + 1$.

   (ii) We wish to get cyclic vector spaces of length 7 over $\mathbb{F}_2$. For this we need to factorise $x^7 - 1$ over $\mathbb{F}_2$. We have
   $x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = (x-1)(x^3 + x^2 + 1)(x^3 + x + 1)$.
   The cubic polynomials are irreducible over $\mathbb{F}_2$ as they have no root in $\mathbb{F}_2$. (Note that these two polynomials are reciprocal polynomials of each other.) Thus there are 3 distinct irreducible factors of $x^7 - 1$ over $\mathbb{F}_2$. Hence, the number of cyclic vector spaces of length 7 over $\mathbb{F}_2$ is $2^3 = 8$.

1.8 Number of cyclic vector spaces over a finite field: Cyclotomic cosets

Let $F = \mathbb{F}_q$ be a finite field with q elements, where q is a power of a prime p. We wish to factorise $x^n - 1$ over F. Assume that $(n, q) = 1$. Then by Euler's Theorem, $q^{\phi(n)} \equiv 1 (\text{mod } n)$, where $\phi(n)$ denotes Euler's totient function. Let m be the least positive integer such that $q^m \equiv 1 (\text{mod } n)$, i.e. $n|(q^m - 1)$. Thus m is the order of q mod n. Then the finite field $\mathbb{F}_{q^m}$ of order q^m contains the group $\mathbb{F}^*_{q^m}$ of nonzero elements as a cyclic group of order $q^m - 1$. Since $n|(q^m - 1)$, $\mathbb{F}^*_{q^m}$ has a unique cyclic subgroup of order n, which consists of the n distinct n-th roots of 1 in $\mathbb{F}_{q^m}$. Thus $\mathbb{F}_{q^m}$ has these n n-th roots of 1, say $\xi, \xi^2, \ldots, \xi^n = 1$, where ξ is a generator of the cyclic group of order n. ξ is also called a primitive n-th root of 1 in $\mathbb{F}_{q^m}$. In this field, $x^n - 1$ factorises as $\prod_{i=0}^{n-1}(x - \xi^i)$. The Galois group of $\mathbb{F}_{q^m}/\mathbb{F}_q$ is cyclic of order m, generated by the Frobenius automorphism $a \mapsto a^q$. Every element a in $\mathbb{F}_{q^m}$ has Galois conjugates $a, a^q, \ldots, a^{q^{m-1}}$. Here $a^{q^m} = a$. Some elements in $\mathbb{F}_{q^m}$ have all these m Galois conjugates as distinct, whereas others have first M_a distinct and they repeat m/M_a times. The product $\prod_{i=0}^{M_a-1}(x - a^{q^i})$ is the minimal polynomial of a (and also of all conjugates of a) over $\mathbb{F}_q$. If a is an n-th root of unity in $\mathbb{F}_{q^m}$, then this product is an irreducible factor of $x^n - 1$ in $\mathbb{F}_q$.

If $0 \le u \le q^m - 2$, let m_u be the least positive integer such that $uq^{m_u} \equiv u (\text{mod } (q^m - 1))$. Then $C_u = \{u, uq, \dots, uq^{m_u - 1}\}$ is called a cyclotomic coset of u for the field extension $\mathbb{F}_{q^m}/\mathbb{F}_q$. If γ is a generator of $\mathbb{F}^*_{q^m}$, and $a = \gamma^u$, then $\{\gamma^i | i \in C_u\}$ is the set of distinct Galois conjugates of a. Observe that m_u is the order of q mod $(q^m - 1)/(u, q^m - 1)$ and it is a divisor of m.

If a is any n-th root of unity in $\mathbb{F}_{q^m}$, then a is a power of ξ, say $a = \xi^s$ for some $0 \le s \le n-1$. Let the number M_a of the distinct conjugates of a be denoted by μ_s, so as to note the dependence on s. Then $M_a = m_u = \mu_s$. Thus if $a = \xi^s$, the conjugates of a in $\mathbb{F}_{q^m}$ are ξ^t where t varies over the set $C_s = \{s, sq, sq^2, \dots, sq^{\mu_s - 1}\}$. Here, since $\xi^n = 1$, the conjugates of $a = \xi^s$ can be obtained just by considering t (mod n). Note also that μ_s can be alternatively obtained as the least positive integer such that $sq^{\mu_s} \equiv s(\text{mod } n)$, so it just depends upon s, q and n. μ_s is also the least positive integer such that $q^{\mu_s} \equiv 1(\text{mod } n/(s, n))$.

The set $C_s (0 \le s \le n - 1)$, where the elements are considered mod n, is called as a cyclotomic coset mod n. Observe that the cyclotomic coset mod n of s coincides with the cyclotomic coset mod n of any element of the coset, i.e. $C_s = C_{sq^i}, 0 \le i \le \mu_s - 1$. The degree of the minimal polynomial of $a = \xi^s$ is equal to $|C_s|$. Any two cyclotomic cosets mod n are either equal or disjoint and the set $\{0, 1, 2, \dots, n-1\}$ is a disjoint union of the cyclotomic cosets mod n. The number of cyclotomic cosets mod n is the number of distinct irreducible factors of $x^n - 1$ in $\mathbb{F}_q$ and $x^n - 1$ is the product of these irreducible factors. The number of cyclic spaces is 2^M where M is the number of cyclotomic cosets mod n for the field $\mathbb{F}_q$. If we want the actual cyclic spaces we need to calculate the irreducible factors of $x^n - 1$ corresponding to the cyclotomic cosets mod n, and use them to find all divisors of $x^n - 1$. The irreducible factor corresponding to C_s is $\prod_{i=0}^{\mu_s - 1}(x - \xi^{sq^i})$ which simplifies to a polynomial over $\mathbb{F}_q$.

Remark. If a is a primitive n-th root of unity and $a = \xi^s$, then $(s, n) = 1$, so $M_a = \mu_s =$ the order of q mod n. Thus any two n-th roots of unity in $\mathbb{F}_{q^m}$, whether conjugate in $\mathbb{F}_{q^m}/\mathbb{F}_q$ or not, must have minimal polynomials over $\mathbb{F}_q$ of the same degree = order of q mod n. The cyclotomic polynomial $\Phi_n(x)$ of order n, which is the minimal polynomial over $\mathbb{Q}$ of a primitive complex n-th root of unity in $\mathbb{C}$, is a monic polynomial in $\mathbb{Z}[x]$, and we consider this polynomial mod p (recall that q is a power of a prime p). By induction we see that the primitive n-th roots of 1 in $\mathbb{F}_{q^m}$ are the roots of this polynomial reduced (mod p) of degree $\phi(n)$. This polynomial factorizes over $\mathbb{F}_q$ as a product of irreducible polynomials, each of degree = order of q mod n, the number of irreducible factors over $\mathbb{F}_q$ being $\phi(n)/(\text{order of } q \text{ mod } n)$.

If $a \in \mathbb{F}^*_{q^m}$ is an n-th root of unity, and with γ a generator of $\mathbb{F}_{q^m}$ and ξ a primitive n-th root of unity in $\mathbb{F}^*_{q^m}$, $a = \gamma^u = \xi^s$, we have $M_a = m_u = \mu_s = |C_u| = |C_s \text{ mod } n| =$ the degree of the minimal polynomial of a and this is further = order of q mod n if a is the primitive n-th root of 1 in $\mathbb{F}_{q^m}$.

From the above discussion we get:

(i) If char.$F = 0$, F has a copy of $\mathbb{Q}$ and so the minimum number of cyclic subspaces of length n in such a field F is $d(n)$, the number of divisors of n (attained for $\mathbb{Q}$). For fields, the number of cyclic subspaces of length n monotonically increases with F and attains the maximum number 2^n when the field F contains all the n n-th roots of 1.

(ii) If char.$F = p > 0$, then p is a prime and F contains a copy of $\mathbb{F}_p$. Suppose $(n, p) = 1$. The number of irreducible factors of $x^n - 1$ over $\mathbb{F}_p$ is $k = \sum_{d|n} \frac{\phi(d)}{\text{order of } p \text{ mod } d} \ge d(n)$ and the number of cyclic spaces over $\mathbb{F}_p$ is 2^k. This number monotonically increases to the upper bound 2^n when the field F contains all the n n-th roots of 1. If $p \equiv 1 \pmod{n}$, then the order of p mod n is 1 and so also mod d for any d dividing n. Hence the number of cyclic spaces of length n for $\mathbb{F}_p$ itself becomes 2^n as $\sum_{d|n} \phi(d) = n$. Alternatively as $p \equiv 1 \pmod{n}$, $n|(p-1)$, so $\mathbb{F}_p$ contains all the n n-th roots of 1, so the maximum number 2^n is attained for $\mathbb{F}_p$ itself. Thus, for $p \equiv 1 \pmod{n}$, for all fields with characteristic p, the number of cyclic spaces of length n over F is 2^n.

(iii) Let char.$F = p > 0$ and $(n, p) = 1$. We know that $\mathbb{Z}_n^*$ is a cyclic group if and only if n is a power of an odd prime, or 2 times a power of an odd prime, or 2 or 4. Suppose the order of $p \pmod{n}$ is $\phi(n)$. For example, the order of 2 mod 9 is $6 = \phi(9)$. In this case one observes that for every divisor d of n, the order of $p \pmod{d}$ is $\phi(d)$. Hence the number of cyclic subspaces of length n over $\mathbb{F}_p$ is 2^k where $k = \sum_{d|n} \frac{\phi(d)}{\phi(d)} = \sum_{d|n} 1 = d(n)$. This is exactly what happens over $\mathbb{Q}$. In this case the factorisation of $x^n - 1$ into irreducible factors over $\mathbb{F}_p$ just comes from the one over $\mathbb{Q}$, that is, $x^n - 1 = \prod_{d|n} \Phi_d(x)$ is the complete factorisation of $x^n - 1$ over $\mathbb{F}_p$.

(iv) If $\mathbb{Z}_n^*$ is not a cyclic group, then p cannot have order $\phi(n)$. Thus the cyclotomic polynomial $\Phi_n(x)$ considered mod p is reducible over $\mathbb{F}_p$. If $\mathbb{Z}_n^*$ is a cyclic group, $\Phi_n(x)$ remains irreducible over $\mathbb{F}_p$ if and only if the order of p mod n is $\phi(n)$, and in that case for every d dividing n, $\Phi_d(x)$ is irreducible over $\mathbb{F}_p$ and $x^n - 1$ is a product of $d(n)$ irreducible factors over $\mathbb{F}_p$.

We write this discussion in the form of

Theorem 4. *The cyclotomic polynomial $\Phi_n(x)$ remains irreducible mod p if and only if n is a power of an odd prime, 2 times a power of an odd prime, or 2 or 4, and the order of p mod n is $\phi(n)$. Also, in this case $x^n - 1 = \prod_{d|n} \Phi_d(x)$ is the complete factorisation of $x^n - 1$ over $\mathbb{F}_p$.*

Thus for $(n, p) = 1$, the number of cyclic spaces over $\mathbb{F}_p$ of length n is $\geq d(n)$ and equality holds if and only if $\mathbb{Z}_n^*$ is a cyclic group and the order of p in this group is $\phi(n)$.

I thank Vaidehee Thatte for querying about varying nature of number of cyclic spaces over fields w.r.t. order relation.

Example 1. The cyclotomic polynomial $\Phi_7(x) = x^6 + x^5 + \cdots + x + 1$ is reducible over $\mathbb{F}_2$, because the order of 2 mod 7 is 3, and $\Phi_7(x)$ factorises over $\mathbb{F}_2$ as a product of 2 irreducible polynomials over $\mathbb{F}_2$ of degree 3.

Example 2. The polynomial $\Phi_9(x) = x^6 + x^3 + 1$ is irreducible over $\mathbb{F}_2$, because the order of 2 mod 9 is $6 = \phi(9)$.

Example 3. If p is a prime and $p|n$, write $n = p^k N$, where $(p, N) = 1$. We want to find the number of cyclic vector spaces of length n over $\mathbb{F}_q$, q being a power of the prime p. Here one first finds cyclotomic cosets mod N for the field $\mathbb{F}_q$ and the corresponding irreducible factors of $y^N - 1$. If $g(y)$ is such an irreducible factor, replace y by x^{p^k}. Then write $g(x^{p^k})$ as p^k-th power of an irreducible polynomial in x, say $g_0(x)$. For this just replace in $g(x^{p^k})$, each coefficient by its (unique) p^k-th root and replace x^{p^k} by x. This is an irreducible polynomial in $\mathbb{F}_q$. The number of cyclic vector spaces of length n is $(p^k + 1)^M$ where M is the number of cyclotomic cosets mod N obtained for $\mathbb{F}_q$. The actual cyclic spaces can be obtained from the irreducible factors of $x^n - 1$, which in turn give us all the divisors of $x^n - 1$.

Example 4. In Example 4(ii) in Section 1.7, $F = \mathbb{F}_2$. To factorise $x^7 - 1$, consider the cyclotomic cosets mod 7: $C_0 = \{0\}$, $C_1 = \{1, 2, 4\}$, $C_3 = \{3, 6, 5\}$. Thus $x^7 - 1$ factorises over $\mathbb{F}_2$ as a product of 3 irreducible factors, the number of cyclic spaces of length 7 is $2^3 = 8$. For ξ being a primitive 7th root of 1 in $\mathbb{F}_8$, the factor corresponding to C_0 is $x - \xi^0 = x - 1$. The other two factors are $(x - \xi^1)(x - \xi^2)(x - \xi^4)$ and $(x - \xi^3)(x - \xi^6)(x - \xi^5)$. These two factors are going to be $x^3 + x^2 + 1$ and $x^3 + x + 1$, which factor is which depends upon the choice of ξ. In this example note that $x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$ over $\mathbb{Z}$ and going mod 2, this factorisation works over $\mathbb{F}_2$, but the second factor further factorises into 2 irreducible factors over $\mathbb{F}_2$.

Example 5. $F = \mathbb{F}_2$. To factorise $x^9 - 1$, consider the cyclotomic cosets mod 9 given by $C_0 = \{0\}$, $C_1 = \{1, 2, 4, 8, 7, 5\}$, $C_3 = \{3, 6\}$. Thus there are 3 cyclotomic cosets mod 9 and the number of irreducible factors of $x^9 - 1$ over $\mathbb{F}_2$ is 3 and we have $2^3 = 8$ cyclic spaces of length 9 over $F = \mathbb{F}_2$ given by the corresponding divisors. Note that $x^9 - 1$ already factorises over $\mathbb{Z}$ as $x^9 - 1 = (x - 1)(x^2 + x + 1)(x^6 + x^3 + 1)$. Reducing mod 2, this give rise to a factorisation over $\mathbb{F}_2$. This is the factorisation into irreducible factors because from cyclotomic cosets mod 9, we have seen that there are only 3 irreducible factors over $\mathbb{F}_2$.

1.9 Cyclic Codes

A linear code C of length n over a finite field F is a subspace of F^n. If this subspace is a cyclic subspace, then it is called a cyclic code. As we have seen, such a cyclic code is cyclically generated by a single vector v in F^n obtained from a divisor $g(x)$ of $x^n - 1$ over F. If $g(x)$ is of degree k, then v and its cyclic translates (total $n - k$ in number) generate an $(n - k)$-dimensional cyclic code over F. Such codes are important in the theory of error correcting codes used when the messages or pictures are sent through a noisy channel. A linear code C of length n and dimension k is called an $[n, k]$-code. If we have 2 codewords (elements of the vector space $C \subset F^n$), the distance between them is the number of unequal coordinates in the two codewords. The minimum distance d of a code is the minimum of nonzero distances between codewords. The code is now called an $[n, k, d]$-code. The advantage of the minimum distance d is that the code (the subspace) can correct $[\frac{d-1}{2}]$ errors.
From the point of view of Coding Theory, if C is a code of length n over a finite field F, then its orthogonal code denoted by $C^\perp$, consisting of all $w \in F^n$ orthogonal to every $v \in C$ (the dot product $v.w = 0$), is an important subspace of F^n. If C is a cyclic code generated by a polynomial divisor $g(x)$ of $x^n - 1$ over F, let $x^n - 1 = g(x)h(x)$. Then the reciprocal polynomial $h_R(x)$ of $h(x)$ is also a divisor of $x^n - 1$ over F. Now consider the vector v in F^n starting with the coefficients of $h_R(x)$ in increasing powers of x (and rest 0). It can be proved that this vector cyclically generates $C^\perp$, i.e. v and its cyclic translates (total $\deg(g(x))$ in number) generate $C^\perp$. Thus $C^\perp$ is also a cyclic code and is easy to obtain from C. The minimum distance d of C can be obtained from $C^\perp$. See [1] for details.
The Hamming code is a [7,4,3]-code over $\mathbb{F}_2$ and it can correct $(3-1)/2 = 1$ error. Hamming code is an example of a cyclic code of length 7 over $\mathbb{F}_2$ and it comes from an irreducible factor of $x^7 - 1$ over $\mathbb{F}_2$ of degree 3 viz. $g(x) = x^3 + x + 1$ and so its dimension is $7 - 3 = 4$. Hamming code is used in every computer to avoid the possibility of an error where very occasionally 0 and 1 may be interchanged. Other examples of cyclic codes are BCH codes (Bose-Chaudhuri-Hocquenghem Codes), Reed-Solomon codes, and some classes of low-density parity-check codes defined from finite geometries.
BCH codes were developed in 1959/60 and are used in applications such as satellite communications, compact disc players, DVDs, disk drives, USB flash drives, solid-state drives, and two-dimensional bar codes (e.g. QR code or Quick Response code). Reed Soloman codes developed in 1960 have similar applications and are slowly being replaced by BCH Codes. For more information about codes the reader may refer to [1].

Reference

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□□□

2. Proofs of Pythagoras Theorem by Bhaskara II: Myths and Facts

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Abstract

Bhaskaracarya, from the twelfth century, has given in his book *Bijganitam* two proofs of the Pythagoras theorem, one algebraic and another geometric. We present here a comprehensive discussion around these proofs, from an historical perspective. We note that the geometric proof is of *dissection* type. We point out that the algebraic proof and the generalized Pythagoras theorem had been discovered by Thabit-Ibn-Qurra a few centuries earlier. Interestingly, the same analysis and the results are found to be rediscovered by Wallis in the seventeenth century. We conclude the article addressing some misunderstandings. It is shown that (i) what is known as the "Behold Proof", attributed to Bhaskara involves a distortion, and similarly, (ii) it is unjustified to say, as done by some authors, that it is not of Indian origin.

2.1 Introduction

The theorem about the diagonals of right-angled triangles, attributed to the Greek philosopher Pythagoras (ca. 6th century BCE) comes in the following two forms:
Theorem I: *The sum of the squares of the lengths of the sides of a right-angled triangle is equal to the square of the length of its diagonal.*
Theorem II: *The areas of the squares drawn on the sides of a right-angled triangle taken together equal the area of the square drawn on the hypotenuse.*
The theorem is stated in the Baudhayana sulbasutra, the foremost and earliest known Sulba work (ca.8th cent. BCE), in the form of Theorem II; cf. Sen et al., [28 a] - the occurrence pre-dates Pythagoras by a few centuries. The theorem was also known in other ancient civilizations, especially to the Babylonians (ca. 2000 BCE) and the Chinese (ca. 1100 BCE); cf. [18], [19], [36]. Numerous interesting applications are also seen to be made of the theorem in the Sulbasutras; cf. [12], [24], [27]. However, on the issue as to how they proved the theorem, in one or the other form, we do not have any definitive evidence. Nevertheless, it seems unreasonable to believe that they would have made practical applications of these results without having some proof of its validity, though the arguments involved may not have met the requirements of rigor that we expect now. Once the principle was noted the proofs followed in due course.
A straight-forward approach to get a geometrical proof is to look for a design where two smaller squares, each drawn on the sides of a right-angled triangle are shown to fit, after suitable subdivision, in the bigger square drawn on the hypotenuse and vice versa; cf. Hui [18], Katz [19], Wagner [33]. This criterion led to the categorization of various proofs as of *dissection type*.

We shall now discuss the treatment of the theorem by Bhaskaracarya (b. 1114 CE), one of the greatest Mathematicians and Astronomers from India, from the medieval times; he is also known as Bhaskara II to distinguish him from another stalwart with the name Bhaskara, who flourished in the 7th century - we shall refer to our author in short as Bhaskara. In his *Bījagaṇitaṃ*, he has given proofs of the theorem both algebraically and geometrically. The result is also mentioned in his earlier book *Līlavatī*, which deals with arithmetic and mensuration. These books, written by him in the middle of 12th century (1150 CE) were very popular, and served as a major source of learning, for several centuries. The pedagogical cause involved was supported by many commentaries and translations of these books, written in Sanskrit as well as in other regional and foreign languages. Well-known among these are Ganesha Daivajnya's commentary on *Līlavatī* titled *Buddhivilasini* (1545 CE) [23] and Krishna Daivajnya's commentary on *Bījagaṇitaṃ* by name *Navankura* (1650 CE) [1]; it has also been edited under the title *Bija Pallavam*, [7].

Bījagaṇitaṃ describes various algebraic principles and techniques through formulas, followed by examples in the form of exercises. The main text is in verse form, but it is supplemented by the author's own commentary in prose, usually referred to as *Vāsānābhaṣya*; it may be noted that there is no separate commentary by Bhaskara on *Bījagaṇitaṃ*. We present in the next section an analysis of the proofs; we refer to [13] for the original verses - among the available editions of this book most include *Vāsānābhaṣya*, but some give only the verse part. Besides, editor-cum-authors also provide their own commentaries. For details, the reader may refer to the recent and exhaustive work by Hayashi [16]. In Section 3 we discuss the proofs in an historical perspective. We conclude the article by addressing some misunderstandings concerning these proofs.

2.2 Bhaskara's Proofs of Pythagoras Theorem

2.2.1 *From* बीजगणितं

Bhaskara's proofs of the Pythagoras theorem in बीजगणितं (Algebra) are given with reference to an exercise in the chapter entitled "एकवर्णमध्यमाहरणम्" (Quadratic equations of a single variable). We present below a typescript from the book by Dwivedi [13], followed by a translation; though translations are available in literature, the translation included here is the present author's own.

उदाहरणम् ।
क्षेत्रे तिथिनखैस्तुल्ये दो: कोटी तत्र का श्रुतिः ।
उपपत्तिश्च रूढस्य गणितस्यास्य कथ्यताम् ॥ १२८ ॥

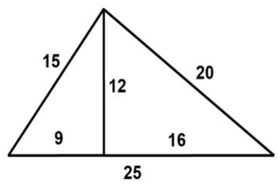

Fig. 2.1

अत्र कर्णः या १ । एतत्त्र्यस्त्रं परिवर्त्य यावत्तावत्कर्णे भूः कल्पिता । भुजकोटी तु भुजौ । तत्र यो लम्बस्तदुभयतो ये त्र्यस्त्रे तयोरपि भुजकोटी पूर्वरूपे भवतः । अतस्त्रैराशिकम् । यदि यावत्तावति कर्णे अयं १५ भुजस्तदा भुजतुल्ये कर्णे क इति लब्धं भुजः स्यात् । सा भुजाश्रिताबाधा रू २२५/ या१ । पुनर्यदि यावत्तावतिकर्णे इयं २० कोटिस्तदा कोटि २० तुल्ये कर्णे केति जाता कोट्याश्रिताबाधा रू ४००/ या १ । आबाधायुतिर्यावत्तावत्कर्णसमा क्रियते तावत्तद्भुजकोटिवर्गयोगस्य पदं कर्णमानमुत्पद्यते २५ । अनेनोत्थापितापितेजाते आबाधे ९ । १६ । अतो लम्बः १२ । न्यासः

Exercise: What is the *hypotenuse* (श्रुतिः) of a right-angled triangle whose *side* (दो:) and *upright* (कोटी) are respectively 15 (तिथि) and 20 (नखै:). Give *also the proof* (उपपत्तिश्च) of the *traditional theorem* (रूढस्य गणितस्य) in this regard[1].

अथान्यथा वा कथ्यते- कर्णः या १ । दोःकोटिघातार्धं त्र्यस्त्रक्षेत्रस्य फलम् १५० । एतद्विषमत्र्यस्त्रचतुष्टयेन कर्णसमं चतुर्भुजं क्षेत्रमन्यत्कर्णज्ञानार्थं कल्पितम् । न्यासः
एवं मध्ये चतुर्भुजमुत्पन्नम् । अत्र कोटिभुजान्तरसमं भुजमानम् ५ ।
अस्य फलम् २५ । भुजकोटिवधो द्विगुणस्त्र्यस्त्राणां चतुर्णामेतद्योगः ६०० ।
सर्वं बृहत्क्षेत्रफलम् ६२५ । एतद्यावत्तावद्वर्गसमं कृत्वा लब्धं कर्णमानम् २५ ।
यत्र व्यक्तस्य न पदं तत्र करणीगतः कर्णः ।

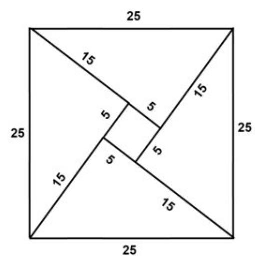

Fig. 2.2

The hypotenuse to be determined is represented as या १[2]. Here, the triangle is considered by turning it around, in a manner so that its *hypotenuse* (यावत्तावत्कर्ण) becomes the base. The side and the upright are the sides. The sides and the uprights of the triangles lying on the two sides of the perpendicular (drawn from the vertex) *are similar to the original right-angled triangle* (पूर्वरूपे भवतः।). *Hence, here is a case of proportion* (अतस्त्रैराशिकम्). If for यावत्तावत् hypotenuse the (corresponding) side is 15 then for the hypotenuse being equal to this side what would now be the side. This (the

[1]As special the format of the question so is the approach outlined by the Master. It is to keep in mind that Bhaskara's objective has been to outline the approach rather than to provide a formal presentation/ discussion.

[2]This is also referred in the sequel as यावत्तावत्कर्ण and vice versa.

answer to this question) will yield the side of a (smaller) triangle; *the projection of the side (of the original triangle)*, भुजाश्रिताबाधा as रू २२५/ या १. Similarly, if for यावत्तावत् hypotenuse the upright is 20 then what would be the upright if the hypotenuse is 20? It would give the value of the projection of the upright (of the original triangle) (कोट्याश्रिताबाधा) as रू ४००/ या १. *Equating the sum of these projections to* यावत्तावत्कर्ण (या १) *and simplifying* (आबाधायुतिर्यावत्तावत्कर्णसमा क्रियते) *one obtains the value of the hypotenuse as equal to square root of the sum of the squares of its sides* (भुजकोटिवर्गयोगस्य पदं कर्णमानमुत्पद्यते ।[3]), 25. *This result used in reverse yields* (अनेनोत्थापितापिते जाते) the values of the projections as 9 and 16 and accordingly, the perpendicular comes out to be 12. See Fig. 2.1.

Bhaskara continues further and gives *another method* (अथान्यथा वा कथ्यते) for evaluating the size of the hypotenuse. The hypotenuse is taken as या १. *The area of the triangle is equal to the half of the product of the side and the upright*, 150 (दोःकोटिघातार्धं त्र्यस्रक्षेत्रस्य फलम् १५० ।). *In order to determine the value of the hypotenuse, we construct* (क्षेत्रमन्यत्कर्णज्ञानार्थं कल्पितम्), *a square of side equal to the hypotenuse* (कर्णसमं चतुर्भुजं) *using four triangles identical to the given triangle* (एतद्विषमत्र्यस्रचतुष्टयेन), See: (Fig. 2.2). In the middle of the figure there arises a quadrilateral, (obviously it will be a square) of *size equal to the difference of the upright and the side of the triangle* (कोटिभुजान्तरसमं भुजमानम्), 5. Its area is 25. The area of the four triangles taken together will be *twice the product of the upright and the side*, (भुजकोटिवधो द्विगुणस्त्र्यस्राणांचतुर्णामेतद्योगः), 600. The total area of the big square comes out to be 625. *Equating this to the square of* यावत्तावत्, *the value of the hypotenuse is obtained as* 25 (एतद्यावत्तावद्वर्गसमं कृत्वा लब्धं कर्णमानम् २५). Bhaskara says that in case the sum is not a perfect square the hypotenuse would be a surd root (करणीगतः कर्णः).

एतत्करणसूत्रं वृत्तम्---

दोःकोट्यन्तरवर्गेण द्विघ्नो घातःसमन्वितः
वर्गयोगसमःसस्याद्वयोरव्यक्तयोर्यथा ।। १२९ ।।

अतो लाघवार्थं दो:कोटिवर्गयोगपदं कर्णं इत्युपपन्नम् । तत्र तान्यपि
क्षेत्रस्य खण्डानि अन्यथा विन्यस्य दर्शनम् ।

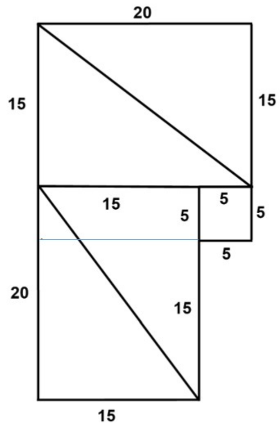

Fig. 2.3

Hereafter, Bhaskara adds that *in this regard there is a formula* (एकएतत्करणसूत्रं वृत्तम्): A square having its sides equal to the difference of upright and the side (of a right-angled triangle) when gets added to twice of their product, (which equivalently means to two rectangles) is equal to the sum of their squares separately; similar to the algebraic relation for two quantities;

$$(a-b)^2 + 2ab = a^2 + b^2.$$

Thus, in essence (अतो लाघवार्थं), *it is proved that the square root of the sum of the squares of the side and the upright (of a right-angled triangle) is equal to the hypotenuse* (दो:कोटिवर्गयोगपदं कर्णं इत्युपपन्नम् ।). Bhaskara states further that *the formula quoted here may be understood by constructing the figures involved; as has been depicted in the figure for the case under discussion or may be done similarly in other cases* (तत्र तान्यपि क्षेत्रस्य खण्डानि अन्यथा विन्यस्य दर्शनम् ।).

2.2.2 Comments and Discussion

(a) Verse 128 / रूढस्य गणितस्य

In the opening two verses of the chapter entitled क्षेत्र व्यवहार (Operations with Plane Figures) of Līlavatī, [2a] Bhaskara states:

[3]भुजकोटिवर्गयोगस्य पदं कर्णमानम्" is equivalent to the mathematical expression: hypotenuse $= \sqrt{[(\text{side})^2 + (\text{upright})^2]}$ Bhaskara referred this as रूढ गणित (traditional theorem) and is what is known as Pythagoras theorem for the square of the hypotenuse.

इष्टो बाहुर्यः स्यात् तत्स्पर्धिन्यां दिशीतरो बाहुः । त्र्यस्रे चतुरस्रे वा सा कोटिः कीर्तिता तज्ज्ञैः ।। १३५ ।।
तत्कृत्योर्योगपदं कर्णो दोःकर्णवर्गयोर्विवरात् । मूलं कोटिः कोटिश्रुतिकृत्योरन्तरात् पदं बाहुः ।। १३६ ।।

The sides which are normal to each other are referred as बाहु (side) and कोटि (upright). (It is said explicitly elsewhere in *Līlavatī* that there is no discrimination as such. Their usage is a matter of preference and is relative to each other). These statements are given with reference to either a right-angled triangle or to a rectangle. It is stated, besides their other interrelations, that तत्कृत्योर्योगपदं कर्ण which means that the square root of the sum of squares of the sides equals कर्ण (hypotenuse/ diagonal) - the well-known Pythagoras Theorem. Bhaskara says that this relation *is known to learned persons* (कीर्तिता तज्ज्ञैः).

Interestingly, a similar statement is also found in the Manava sulbasutra (5th cent. BCE): आयामम् आयामगुणम् विस्तारं विस्तरेण तु । समस्य वर्गमूलम् यत् तत्कर्णम् तद्विदो विदुः ।।१०.१० ।।), Sen[28 b], (Multiply the length by the same length and the breadth by the breadth; the square root of the sum of these two (results) gives the hypotenuse; this is already *known to the scholars* (तद्विदो विदुः)).

It seems that by the time of Bhaskara there would have been a standard (or traditional) proof of the theorem, passed on from teacher to student, though not recorded, which is what Bhaskara alludes to as रूढ गणित.

(b) एतद्विषमत्र्यस्रचतुष्टयेन कर्णसमं चतुर्भुजं क्षेत्रमन्यत्कर्णज्ञानार्थं कल्पितम् ।

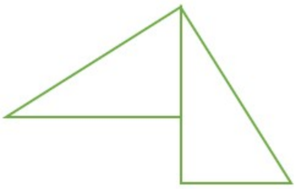

Fig. 2.4

It is stated that a square is to be constructed using four equal right-angled triangles. Krishna Daivajnya in his commentary of Bijaganitam has elucidated it step by step, [1]. To begin with, one of the triangles is to be turned around by a right-angle and then depending on the situation its side/upright is placed in contact with the upright/side of another triangle. We display here the first step and the rest follow in a similar manner (see Fig. 2.4). The resulting figure will be a square (and not just a rhombus), with a "square hole" in between, as seen in Figure 2.2.

(c) Geometrical proof as dissection type

The geometrical proof, through a change of perception, may be understood as of dissection type. One starts with a square with side equal to the hypotenuse of the right-angled triangle; then dissects it into four identical right-angled triangles with a left over as a square of the size equal to the difference of the side and the upright. Now one arranges them as shown in Fig. 2.3. A second dissection is needed to get it into two squares each of sizes equal to the sides of the right-angled triangle. This completes the proof.

(d) दोःकोट्यन्तरवर्गेण द्विघ्नो घातः समन्वितः। वर्गयोगसमः स स्याद् द्वयोरव्यक्तयोर्यथा ।। १२९ ।।

This is essentially a proposition in Geometry regarding the squares of sides and the difference of the side and the upright of a right-angled triangle. As we discussed earlier this proposition is analogous to the algebraic identity: $(a-b)^2+2ab = a^2+b^2$. Following to this proposition Bhaskara gives the statement: "तत्र तान्यपि क्षेत्रस्य खण्डानि अन्यथा विन्यस्य दर्शनम्", supplemented by a figure (Fig. 2.3). Through this statement Bhaskara says that one may draw a similar figure to prove (see) the validity of the proposition.

2.3 Bhaskara's proofs in Historical Perspective

Colebrooke, a prominent Sanskrit Scholar who spent over 30 years in India, has made an extensive study of contributions of medieval Indian mathematicians through which he brought out the essence of the works of Aryabhata, Brahmagupta, Bhaskaracarya, and of many others, to the western world. He has made an analysis of *Bījagaṇitaṃ* and *Līlavatī*, [8].

2.3.1 *Algebraic Demonstration*

Colebrooke sensed immediately the relevance of Bhaskara's investigations. He, in the chapter entitled "Dissertation on the Algebra of Hindus with notes and Illustrations, [9] writes: ".... *The demonstration of the noted proposition of Pythagoras ...The demonstration is given in two ways in Bhaskara's Algebra (Vij-gan. §146). The first of them is the same as delivered by Wallis in his treatise on the angular sections (Ch. VI) and, as far as appears, then given for the first time*".

(a) Wallis' Analysis:

(i) Angle $A = 90°$

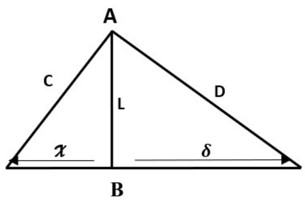

Fig. 3.1

John Wallis, [34] designates sides as C, D; base B and Segments χ, δ. Let L stands for the perpendicular from the vertex A to the hypotenuse, which is now base of the triangle; refer to Fig. 3.1. Triangles CBD, $CL\chi$ and $DL\delta$ are similar to each other. A straight-forward analysis based on the principle of proportions leads to $C^2 + D^2 = B^2$

Thus, Pythagoras theorem is proved. In Bhaskara's notation B, C, D, χ, δ and L are कर्ण, भुज, कोटि, भुजाश्रिताबाधा, कोट्याश्रिताबाधा and लम्ब respectively. Wallis writes: "*The noted proposition of Pythagoras (which is in Euclid, 47 Book I) ...gave me occasion to pursue that speculation a little further...*". This matter is closely related to proposition 8, Book 6 of Euclid's Elements; similar figures: "*If, in a right-angled triangle, a (straight line) is drawn from the right-angle perpendicular to the base then the triangles around the perpendicular are similar to the whole (triangle), and to one another*" (see, Fitzpatrick, [15]). As a Corollary to this, it is stated: "*If, in a right-angled triangle, a (straight line) is drawn from the right-angle perpendicular to the base then the (straight line so) drawn is in mean proportion to the pieces of base.*" This means $L = \sqrt{\chi\delta}$. It is interesting to note that Euclid was very close to the proof but he did not spell it out. This proof was unknown in Europe till Wallis rediscovered it, says Cajori, [5, 6 b].

(ii) Arbitrary Angle

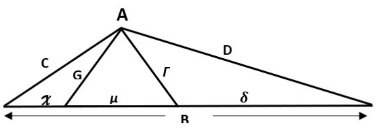

Fig. 3.2

Wallis, in continuation of the above considers the particular case, $A = 120°$. Two straight lines G and Γ are drawn to the base making angles with the base equal to that at the top, see Fig. 3.2. The two triangles $GC\chi$ and $\Gamma\delta D$ are like to the whole CDB. The triangle, $G\Gamma\mu$ lying in between is equilateral. Proceeding as before it is easy to show that $C^2 + D^2 + CD = B^2$.

It is thus stated: "The square of the base (of an angle $120°$*) is equal to the squares of the legs and a rectangle of them."*

Wallis now takes up in succession the cases, $A = 60°, 135°, 45°$ and finally states the general result for arbitrary value of A.

"That is, The square of the base (whatever be the angle at the vertex) is equal to the squares of the legs, together with (if it be greater than a Right-angle) or wanting (if less than such) a plain, which shall be, to the rectangle of the legs, as a portion in the base-line, intercepted between two lines from the vertex, making at the base a like angle with that of the vertex, to one of those two lines so drawn." i. e. $C^2 + D^2 \pm [CD\mu/G] = B^2$; the upper sign is for A being obtuse. This relation may also be expressed equivalently as: $C^2 + D^2 \pm [B\mu] = B^2$.

(b) Thabit-Ibn-Qurra's Analysis

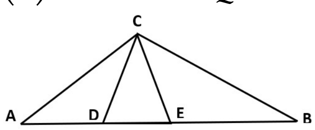

Fig. 3.3

Thabit-Ibn-Qurra, an important mathematician of earliest 9th century CE, had already done the general analysis. Strick (2012) has given an account of his works, [22, 31]. Thabit has given some new proofs of the Pythagoras Theorem. In one of the proofs, he has provided a generalization of the Pythagoras theorem, refer Sayili, (1960), [26]. Let in a triangle ABC the points D and E are drawn on the base AB in such a way that the triangles ABC, ACD and CBE are similar to each other. It will require the lines CD and CE to be drawn such that these make angles with base equal to the angle at the top, C. This

prescription is equivalent to what has been given by Wallis. Obviously, the triangle, CED lying in between is an isosceles triangle, see Fig. (3.3.). Now applying the principle of proportion, it is demonstrated that $AC^2 + BC^2 = AB(AD + EB)$. It has been established further that this relation yields: $AC^2 + BC^2 = AB^2 \pm (AB \times DE)$; the upper sign is for the case C is acute. It is pointed out that in some situations either or both of the points D and E may require to be taken on the extended line AB. It is evident that this relation is equivalent to that obtained by Wallis.
Thus, Thabit-Ibn-Qurra is credited for the pre-discovery of results obtained by Wallis. Further, it is simple to see that this relation is equivalent to the trigonometric identity: $B^2 = C^2 + D^2 - 2CD\cos A$. In the case $A = 90°$ the two lines G and Γ coalesce as one, leading to $\mu = 0$. Accordingly, one gets the Pythagorean relation. In conclusion, the algebraic demonstration of the Pythagoras theorem may be put in the time line as: Thabit (9th cent. CE), Bhaskara (12th cent. CE), Wallis (17th cent. CE). It is to be noted that Thabit is known to have translated Euclid's Elements. Hence the source of the works of Thabit and of Wallis both may be attributed to Euclid's Elements.

2.3.2 Geometric Demonstration

(a) Chou Pei Suan Ching, an Ancient Chinese Book
A figure known as Hypotenuse Diagram is cited in Chou Pei Suan Ching; the oldest of Chinese mathematical classics (c. 1100 BCE). This diagram is similar to Figure 2.2. but as we will discuss in the sequel, there is a minute but crucial difference to note. We reproduce here this diagram from Needham and Ling, [21 b], wherein it is discussed with reference to an ancient dialogue between Chou Kung (the duke of Chou) and a personage named Shang Kao.

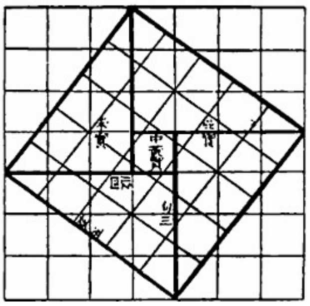

Fig. 3.4 The proof of the Theorem in the Chou Pei Suan Ching

"(3). *The rectangle originates from (the fact that)* $9 \times 9 = 81$ *(i.e. the multiplication table or the properties of numbers as such).*
(4). Thus, let us cut a rectangle (diagonally) and make the width 3 (units) wide and the length 4 (units) long. The diagonal between the (two) corners will then be 5 (units) long. Now after drawing a square on the diagonal, circumscribe it by half- rectangles like that which has been left outside, so as to form a (square) plate. Thus, the (four) outer half-rectangles of width 3, length 4 and diagonal 5, together make two rectangles (of area 24); then (when this is subtracted from the square plate of area 49) the remainder is of area 25. This (process) is called "piling of rectangles".
Here, a square on the hypotenuse of a right-angled triangle of sides 3, 4, 5; and a subscribing square plate are drawn (see Fig. 3. 4). Thereafter, area of the square on hypotenuse is evaluated as the residual of the square plate and the subscribing area, $7^2 - 4(3 \times 4)/2$ leading the result as 25. It is to be noted that here concern is the region in between the square plate area and the hypotenuse square. Of course, triangles and a smaller square are seen prominently lying within the hypotenuse square. *Bhaskara considers the hypotenuse square embodied with triangles and squares.* Later writers, notably Zhao Shuang, ([36], [11c]), Hui [18] switched their attention from the square plate area to the hypotenuse area; and it seems that since then the Hypotenuse diagram is being meant as the hypotenuse square area endowed with triangles and a square. Needham et al., [21c] writes that in the time of Liu and Chao; (cited as 3rd cent. CE), the central small square was yellow and the surrounding rectangles red.
Needham and Ling, [21a] in their introductory remark to the dialogue, write: *"From a dialogue...on the properties of the right-angled triangle in which the Pythagorean theorem is stated, though not proved in the Euclidean way"*. We will discuss this issue further in the next section; interested readers may consult Cullen [10], Katz [19], Swetz and Katz [32], Wagner [33]. Heath (1908), [17c], making a reference to Zeuthen (1904), has discussed how the hypotenuse diagram may be considered as a proof of the Pythagoras theorem for right angled triangles of rational sides. Bretschneider's conjecture for the procedure through which Pythagoras might have got to the proof is well known. This method is based primarily on the hypotenuse diagram. The proposal is to consider a square

of the size equal to the sum of the sides, a and b and thereafter to make use of the algebraic identity, $(a+b)^2 = a^2 + b^2 + 2ab$; (of course, as given by Euclid) (see, Heath [17 a], Eves [14 a], Cajori [5], [6 b], Burton [4 a]). *This analysis is analogous counterpart of that given by Bhaskara.*

(b) Aryabhatiyabhasya, a commentary of Aryabhatiya

Srinivas, [30 b] points out that Bhaskara I (c. 629 CE) has employed a figure similar to Fig. 2.2 in his book, *Aryabhtiyabhasya*. This work is a commentary on Aryabhatiya, the famous composition of Aryabhata. Aryabhata, in the Mathematics Section, verse No. 3 of Aryabhatiya discusses the square- both the shape and the area. Bhaskara I, in his commentary to this gives an exercise employing a figure, which may be considered *as a composite of both, adopted by ancient Chinese and by Bhaskara II.* It is to note that the objective for this exercise, as explained by Bhaskara I himself, was to let the reader fully be clear about the concepts of the area and of the square. I reproduce below the diagram along with my English translation of the related section; based on the edited version by Shukla, [29].

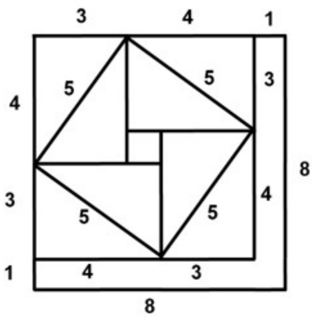

Fig. 3.5

Hence, we conceive of a square of 8 in which four rectangles each of sizes 3 and 4 along with their hypotenuses, 5 are also drawn. In such a conceived figure of the rectangles there arise a centrally situated square lying within the square of sides equal to the hypotenuse. The area of the square formed with these rectangles is equal to square of the hypotenuse including to that of the central square. The triangles as half of the rectangles are also seen in the figure. To obtain the areas by two different manners the above diagram is presented. Here, there are various squares whose areas are the squares of their sides. ...visualizing various figures, one should practice these concepts.

2.4 Discussion and Conclusion

Bhaskara gave proofs of the Pythagoras theorem via an exercise for determining the hypotenuse of a right-angled triangle of sides 15, 20. This analysis was immediately understood in its proper perspective. It is sometimes stated that the geometrical proof was given first and then at another place was given the algebraical proof; see Benson [3], Cajori [6 b]. This is not correct. In this context, it is to be noted that Bhaskara gave proofs in *Bījgaṇitaṃ* at one stretch, and the algebraical proof was followed by the geometrical proof.

It is believed that Bhaskara has given the proofs in *Līlavatī. We did not find anything to support this claim.* However, there is following related exercise in *Līlavatī. What is the hypotenuse where the upright is 4 and the side is 3? Tell how one would determine the value of upright from the given side and the hypotenuse; and the side from the given upright and the hypotenuse,* Apte [2 b]. Ganesha in his commentary to this exercise provides the proof of the Pythagoras theorem both algebraically and geometrically, see Ramakalyani [23]. Interestingly, *it is put in the same format and syntax as had been given in Bījagaṇitaṃ*; cf. Srinivas [30 a].

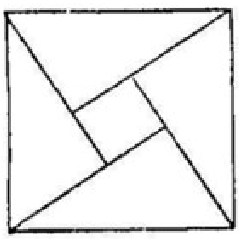

Fig. 4.1

Now we will address two serious misunderstandings, which have been transmitted from one generation to another. These are deep rooted and seem to have originated almost since these originally reached in the West. We will show below that these are devoid of any basis.

(a) Behold proof: The geometrical proof has been glorified as Behold proof. Yannet (1899), [35] making a reference to Cajori's book writes: *"The Hindu Bhaskara, the author of this method, complimented his readers by condensing his proof into the single word, "Behold."* Similar narratives are also seen in the present-day media and elsewhere. In this context, the following citation from *"The Thirteen books of Euclid Elements"* by Heath (1908), [17 b] is worth considering.

"Thus Bhaskara (born 1114 A. D.; see Cantor, I_3, p. 656) simply draws four right-angled triangles equal to the original one inwards, on each side of the square on the hypotenuse, and says "see", without even adding that inspection shows that $c^2 = 4ab/2 + (a-b)^2 = a^2 + b^2$."

Of course, this is a genuine concern. But Bhaskara had already provided the required through verse 129 and it is in this context that he used the word "see", refer Section 2.2 (d) for further details. It appears Fig. 2.3 has not been taken into account, otherwise such remark would not have come out.

Eves, [14 b] writes *"Bhaskara drew the figure and offered no further explanation than the word "Behold". A little algebra, however supplies the proof."* Cajori, ([5], [6 a]) also writes similarly: *"Behold" says Bhaskara without adding another word of explanation"*. Recently, Burton (2011), [4 b] has also written the same thing. These authors have included Fig. 2.3 as part of their discussion, and hence their making such remark seems only a misunderstanding. Sarasvati Amma, [25] cites verse 129 as providing the rationale for the Pythagoras theorem. This is also not correct exactly; as this verse is only a part of the proof. Thus, it seems that the word *"see"* (दर्शनम्) appearing at the end of the statement "तत्र तान्यपि .... विन्यस्य दर्शनम्" has not been understood in its proper context.

(b) "The geometrical proof is not of Indian Origin"

Cajori (1919), [6 b] writes: *"Recently it has been shown that this interesting proof is not of Hindu origin, but was given much earlier (early in the Christian era) by the Chinese writer Chang Chun Ching, in his commentary upon the ancient treatise, the Chou-pei."* (See also [5], [14 b]). Needham et al. (1959), citing a reference to Ling (1956), [20] writes similarly in a footnote ([21 c]): *"it seems extremely probable that Bhaskara's treatment derives from the Chou Pei"*.

In continuation to our earlier discussion on Chou Pei given in Sec. 3.2 (a) we would like to quote, (except the Chinese words) from a recent analysis of *Zhou bi suang* (a Chinese work dated approximately 100 BCE, [36]) by Cullen (1996), [11 a].

".... I must deal with commonly made claim that this section embodies something like a proof of Pythagoras theorem I think that it should be clear from my translation that there is nothing in the main text that could be considered an attempt as proof, unless we make a major effort of imagination followed by heroic emendation. Nor is there anything in the Zhao Shuang's commentary to suggest that he can see a proof in this material. (See for instance Needham (1959), 22-3 and 95-6, followed by van der Waerden (1983), 26-7, also Lam and Shen (1984), 88-9...)"

Cullen writes further at the next page, [11 b]: *"Of course, I do not deny that with sufficient ingenuity you can use the hypotenuse diagram to give a general proof of the Pythagoras theorem by the dissection and re-arrangement of areas. The point is that neither the main text nor Zhao do in fact see it for this purpose.*

In fact, the only evidence we have for anything that could be called an ancient Chinese proof of the Pythagoras theorem appears in the third century commentary of Liu Hui to the Jiu zhang suan shu, near the beginning of chapter nine, Gou gu 'Base and Altitude", which uses the relation repeatedly. Lui is commenting on the first three problems of the chapter, which use the case of base 3. Altitude 4 and hypotenuse 5 to illustrate how each side may be found from the other two. In particular he explains the instructions to find the hypotenuse from the square root of the sum of the squares of the other two sides. ...

All Liu Hui's original diagrams were lost by the Song, and any attempt at restoration must be conjectural.... The point is that whatever diagram he was using, Liu sees no point in giving the reader more than a vague gesture at the process to be carried out, ..."

Besides, as we have discussed earlier in Section 3.2 (b), Bhaskara I (c. 629 CE) has employed a composite figure with the same purpose. All said and done, we conclude that the remark under the title, at (b) above, does not seem to be justified.

Acknowledgement

I am extremely grateful to Prof. S. G. Dani for his encouragement and advice. I am also thankful to Prof. M. S. Sriram for his valuable comments. I also thank Google and Wikipedia. I dedicate this article to my student, (Late) Dr. Rahul Gupta, who has kindly handed over some rare books related to ancient Indian Mathematics from his private collection. I also thank my daughter Dr. Triranjita Srivastava who has assisted me in preparing the diagrams.

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3. What is Happening in the Mathematical World?

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3.1 Partition Functions applied to Detect Prime Numbers

A team of mathematicians has discovered a surprising way to detect prime numbers. They link prime numbers to partition functions, offering fresh ways to detect primes without checking divisibility. The discovery connects two distinct mathematical fields, bridging the gap between combinatorics and number theory with innovative equations.

Ken Ono (left), a top mathematician and advisor at the University of Virginia, and his colleagues William Craig (middle) and Jan-Willem van Ittersum (right) have uncovered an unsuspected link between prime numbers and a completely different mathematical

Ken Ono, a top mathematician and advisor at the University of Virginia, and his colleagues William Craig and Jan-Willem van Ittersum have uncovered an unsuspected link between prime numbers and a completely different mathematical field: integer partitions. This connection could revolutionize our understanding of prime numbers and unveil a hidden pattern in what was once considered pure randomness.

Their findings, published in Proceedings of the National Academy of Sciences (PNAS), earned them a runner-up spot for the 2025 Cozzarelli Prize in the physical sciences. This is the first pure mathematics paper to be recognized in many years.

The biggest known prime is more than 41 million digits long. And there is an infinite number of primes still to find. Identifying patterns among them is one of maths' greatest challenges. That is where Ono's new method comes in.

The idea may sound simple at first. A partition is just a way of breaking a number into a sum of smaller whole numbers. Take the number 5. It can be split in seven ways: $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1$, and $1+1+1+1+1$. But hidden in those combinations are clues. Ono's team showed that partition functions - mathematical tools that count these splits - can be used to detect whether a number is prime. It is remarkable that such a classical combinatorial object - the partition function - can be used to detect primes in this novel way.

For $a > 1$ any positive integer n, the partition function $M_a(n)$ is defined as the sum of the products of the multiplicities of partitions of n with a different part sizes, that is

$$M_A(n) = \sum_{0<s_1<s_2<\cdots<s_a,\ n=m_1s_1+m_2s_2+\cdots+m_as_a} m_1m_2\cdots m_a.$$

Accordingly, $M_1(1) = 1$ and $M_a(1) = 0$ for all $a > 1$; $M_1(5) = 2$ and $M_2(5) = 5$. The following results have been proved in the paper [2].

1. For positive integers n, we have $E_1(n) := (n^2 - 3n + 2)M_1(n) - 8M_2 \geq 0$, and for $n \geq 2$, this expression vanishes if and only if n is prime.

2. For positive integers n, we have $E_2(n) := (3n^3 - 13n^2 + 18n - 8)M_1(n) - (240n - 400)M_2(n) - 960M_3(n) \geq 0$, and for $n \geq 2$, this expression vanishes if and only if n is prime.

By taking suitable linear combinations (with polynomial coefficients) of $E_1(n)$ and $E_2(n)$ as above, it is simple to produce infinitely many such expressions. For example,
$E(n) := (3n^2 + 3)E_1(n) + 2E_2(n) = (3n^4 - 3n^3 - 17n^2 + 27n - 10)M_1(n) - (240n - 400)M_2(n) - 1920M_3(n) \geq 0$ and $E(n) = 0$ if and only if n is prime.
Authors have given 5 such expressions including $E_1(n)$ and $E_2(n)$ and conjectured that:

If $p_1(x), p_2(x), \ldots, p_a(x) \in Z[x]$ be relatively prime integer polynomials. For integers n $E(n) = p_1(n)M_1(n) + p_2(n)M_2(n) + \cdots + p_a(n)M_a(n) > 0$, and vanishes precisely on the primes then $E(n)$ is a $Q[n]$-linear combination of those 5 expressions.
Beyond its intrinsic mathematical interest, this work may inspire further investigations into the surprising algebraic or analytic properties hidden in combinatorial functions. Mathematicians could now explore how these methods might apply to other types of numbers or functions.
Ken Ono is known for building on the legacy of legendary mathematician Srinivasa Ramanujan. Ramanujan's work with partitions a century ago laid the foundation for many ideas explored today.
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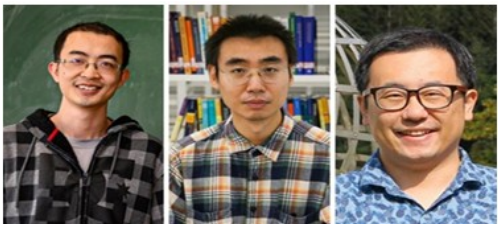
3.2 A 65-Years Old Kervaire Invariant Problem Resolved

In a landmark achievement that has fascinated mathematicians worldwide, a team of Chinese mathematicians has finally resolved the decades-old Kervaire invariant problem, often referred to as the "doomsday hypothesis". The breakthrough was achieved using advanced computational methods, paving the way for future problem-solving in the field. The collaborative effort between Fudan University and UCLA highlights the importance of international cooperation in scientific discoveries.
The Kervaire Invariant Problem is the question of determining the dimensions in which there exists a smooth, framed manifold with a non-zero Kervaire invariant. A framed manifold is a smooth, compact manifold equipped with a framing of its stable normal bundle (or equivalently, its stable tangent bundle).
Framing, in simple terms, is a way to select a field of orthonormal bases for the normal directions everywhere on the manifold. To frame a manifold, it is typically embedded in a high-dimensional Euclidean space $\mathbb{R}^{n+k}$. The normal bundle is the bundle of vectors perpendicular to it at each point.
The Kervaire invariant $k(M^n)$ is an invariant that measures whether an n-dimensional framed manifold M^n can be converted into a homotopy sphere via surgery, a concept introduced by *John Milnor* in 1950.
$k(M^n)$ can only be non-zero if the dimension n is 2 less than a power of 2, i.e., $n = 2^j - 2$ for some integer $j \geq 1$. It takes values in $\mathbb{Z}/2\mathbb{Z}$, meaning it is either **0** or **1**. The Kervaire invariant is a measure of a topological obstruction in surgery theory:

$$k(M^n) = \begin{cases} \mathbf{0} \text{ if } M^n \text{ can be converted into a homotopy sphere} \\ \quad \text{by a sequence of framed surgeries.} \\ \mathbf{1} \text{ otherwise (i.e. the obstruction is non-zero).} \end{cases}$$

When a manifold can be transformed into a sphere, its Kervaire invariant is zero. However, the trick lies in discovering dimensions where the invariant is non-zero, meaning these dimensions can host unusual shapes that defy conversion into a sphere. Smooth framed manifolds with Kervaire invariant 1 are known to exist for $j = 2, 3, 4$ and 5 and corresponding to the dimensions: $n = 2, 6, 14, 30, 62$. And Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel proved in 2009 that for $j > 7$ (i.e., for dimensions n = 254, 510... and all higher cases), no such manifold exists.
The case $j = 6$ (dimension $n = 126$) remained open for many years, earning it the moniker "The Last Kervaire Invariant Problem."

The recent solution by the Chinese team has confirmed that manifolds of Kervaire invariant one do indeed exist in dimension 126, finally concluding that Smooth framed manifolds with Kervaire invariant one exist if and only if the dimension n is 2, 6, 14, 30, 62 or 126.



This monumental discovery was spearheaded by *Wang Guozhen* (middle) and *Lin Weinan* (left) from the Fudan University Shanghai Centre for Mathematical Sciences, in collaboration with *Xu Zhouli* (right) from the University of California, Los Angeles (UCLA). Their paper has already made waves in the academic world for its innovative approach and comprehensive analysis. The team's success was largely attributed to their use of sophisticated computational methods, which allowed them to tackle the problem from a fresh perspective. This opens up new avenues for research, particularly in the study of high-dimensional spaces and their properties. Moreover, the computational techniques developed could be applied to other unresolved problems, potentially accelerating discoveries across various mathematical disciplines. The breakthrough also highlights the evolving nature of mathematics as a field that increasingly intersects with technology. This integration of computation and theoretical analysis represents ashift in how mathematicians approach problem-solving, offering a blend of traditional and modern methodologies that could redefine the landscape of mathematical research.
Source: *https://www.rudebaguette.com/en/2025/07/they-cracked-what-no-one-could-chinese- mathematicians-solve-65-year-kervaire-mystery-and-rock-the-global-math-world/*

3.3 The Mizohata–Takeuchi Conjecture is Disproved by a 17-Year-Old Student

The Mizohata-Takeuchi conjecture was a long-standing problem in harmonic analysis. It proposed a specific weighted inequality for the Fourier extension operator associated with a smooth hypersurface (like a sphere or paraboloid) in Euclidean space.
The conjecture, formulated in the 1980s, essentially stated that for any non-negative weight function ω, the L^2 weighted norm of the Fourier extension operator applied to a function f defined on the hypersurface could be controlled by the L^∞ norm of the X-ray transform of the weight , multiplied by the L^2 norm of f on the hypersurface.
Mathematically, a key formulation of the conjecture is as follows: Let $\Sigma \subset \mathbb{R}^d$ be a compact C^2 hypersurface in $\mathbb{R}^d$ with surface measure $d\sigma$. The extension operator
$\varepsilon : L^2(\Sigma, \mathbf{d}\sigma) \to L^\infty\left(\mathbb{R}^d\right)$ is defined by $[\varepsilon f](x) := \int_\Sigma e^{2\pi i\langle x|\eta\rangle} f(\eta)\mathbf{d}\sigma(\eta)$. The Mizohata-Takeuchi conjecture can be stated as follows: Let $\Sigma \subset \mathbb{R}^d$ be a compact C^2 hypersurface in $\mathbb{R}^d$ with surface measure $\mathrm{d}\sigma$. Let $f \in L^2(\Sigma,\ \mathrm{d}\sigma)$ and let $\omega : \mathbb{R}^d \to \mathbb{R}^+$be a nonnegative weight. Then we have

$$\int_{\mathbb{R}^d} |\mathcal{E}f(x)|^2 w(x)\mathrm{d}x \lesssim \|f\|^2_{L^2(\Sigma;\mathrm{d}\sigma)} \|Xw\|_{L^\infty}$$

where $X\omega$ denotes the X-Ray transform of ω, which is defined as follows. For a function f defined in $\mathbb{R}^n$ (e.g., representing density or attenuation), the X-ray transform, denoted by Xf, is a new function defined on the space of all lines L in $\mathbb{R}^n$: $Xf(L) = \int_L f(x)ds$: where the integral is taken along the line L with respect to the arc length ds.

The conjecture came up as an optional assignment, a small part of a larger homework set to 17-year-old Hannah Cairo at the University of California, Berkeley. Though it was framed as a challenge only meant to stretch students' curiosity, Cairo found herself obsessed on this problem. It was a decades-old mathematical puzzle believed to be true by leading experts in harmonic analysis. For weeks, she explored every angle of the Mizohata-Takeuchi conjecture. After months of failed attempts and relentless thinking, Cairo discovered something unexpected: a counterexample. This means that the conjecture was false!

Cairo's proof used a wide range of tools, including fractals, to arrange her approach precisely. Her insight not only overturned what many in the field assumed was true, but also proved that the structure underlying the conjecture was not as universal as once thought.
After solving the problem, Cairo reworked the proof by transforming it into frequency space - a more abstract mathematical domain where wave behavior becomes clearer. That is when she realized she could build an even simpler counter example. The primary result she proved is as follows.
Primary result (Counterexample): For any C^2 hypersurface $\sum$ that is not a plane, there is some $f \in L^2(\Sigma,\, d\sigma)$ and nonnegative weight $\omega_R : \mathbb{R}^d \to \mathbb{R}^+$so that the following holds.

$$\int_{B_R(0)} |\mathcal{E}f(x)|^2 w(x)\mathrm{d}x \gtrsim \log R \|f\|^2_{L^2(\Sigma;\mathrm{d}\sigma)} \sup_{\ell\subset\mathbb{R}^d \text{ a line}} \int_\ell w.$$

Her work earned her an invitation to speak at the 12th International Congress on Harmonic Analysis and Partial Differential Equations in El Escorial, Spain. The event, organized by the Institute of Mathematical Sciences and the Autonomous University of Madrid, is one of the most respected conferences in the field.
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A Compact Course on Complex Analysis

Region: Gujarat **Dates: September 15-17, 2025**
Institute: Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda
Local Coordinator: Prof. V. O. Thomas, Head, Department of Mathematics.
Regional Co-ordinator: Chirag M. Barasara, Hemchandracharya North Gujarat University, Patan, Gujarat
Resource Person: Prof. V. M. Sholapurkar, Bhaskaracharya Pratishthana, Pune.
Topics: Complex valued functions, Continuity, Differentiability, Analyticity Complex Integration, Zeros and Singularities, Transformations and Conformal Mapping.

In the picture: Professors V. O. Thomas, V. M. Sholapurkar and V. D. Pathak

4. A Peep into History of Mathematics

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In this entry of Peep-series I present a glimpse of two episodes, from 17th and 19th centuries.

Benjamin Wardhaugh, Graphs in the 1680s: Martin Lister, Robert Plot, William Molyneux and John Warner, British Journal for the History of Mathematics, Vol. 38 (2023), No. 2, 97 - 106.

Graphical methods of presentation of data are noted to have acquired public attention, for the first time, near the end of the 18th century, and in particular William Playfair's use of them, in 1786, in displaying economic data is well-known in this respect. However, it turns out that the idea had made its appearance about a hundred years before that, in the 1680's, in recording barometric observations over extended periods, in the form of a line graph, on a background equipped with a grid like on a graph paper. This phenomenon of what is presumably the earliest use of line graphs seems to have been short-lived however, extending just a few years, with nothing like it found for about a century to follow. Furthermore it remained confined to recording of weather.

This brief historical event is the subject of the present paper. The account starts with Martin Lister, a physician and FRS, presenting at the meeting of the Royal Society, London, on 31 October 1683, his method of illustration in which "the account for the whole month [of barometer readings] was but one red line bending as the quick silver [mercury] rises or falls.". The idea is then found proposed, and executed under the aegis of the Oxford Society, by Robert Plot, and his observations for January - December 1684 were printed in *Philosophical Transactions*, dated 23 March 1685; these seem to have been the first printed line graphs. The paper mentions also another similar episode, also from 1685-86, involving William Molyneux, associated with the Dublin Philosophical Society, and also an instrument maker, John Warner from London, who wrote to Plot proposing "... if you desire any more of these [graph] papers you may send for them as at bottom directed for 3s per dozen."! All these developments were apparently inspired by Lister's presentation, though the precise connections are not clear.

The paper features several quotations from the original records and facsimile reproductions of exemplars, of Plot, Molyneux and Warner.

Detlef D. Spalt, The first and most elementary construction of real numbers, Mathematische Semesterberichte 70 (2023), 25 - 41.

Realization of real numbers as Dedekind cuts on the set of rational numbers, is currently viewed as the earliest satisfactory construction from a formal point of view. The concept is noted to have emerged in a rudimentary form in a paper of J. T. Müller in 1858, and in its present form in a paper of Richard Dedekind in 1872; it may be noted that the other familiar notion of real numbers, as the metric completion of rationals, came up only in the 20th century.

It turns out, however, that there is another definition, completely satisfactory in terms of formal presentation, involved in earlier work of Karl Weierstrass. The latter lectured on "Introduction to the theory of analytical functions" fifteen times from the winter of 1865-66 until the summer of 1886, and notes from these, by various redactors, have come to light starting 1988. The historians who studied them did not realize from the earlier sets that they include a formal notion of real numbers. Another manuscript of notes, by Emill Strauss, was discovered in 2016. Based on Struass's notes the present paper describes Weierstrass' definition, which in the title is referred to as the "first and most elementary construction ..."

The construction proceeds along the following steps. To begin with, positive real numbers are constructed as (possibly infinite) *sets of pairs* (m, n), m, n natural numbers, with certain equivalences between the sets (this part is a bit cumbersome, but "elementary" and formal); for such a set S, when the sums of the ratios $\frac{m}{n}$ taken over (m, n) in finite subsets of S are bounded (by a fixed integer), the corresponding real number in our current sense, is the least upper bound of such finite sums; this yields all positive real numbers (the other sets are not needed). Extension to the full set of real numbers is then carried out in the way one extends a semigroup to a group.

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5. Undergraduate Math Competition - A journey from MMC to IMC

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A team of four students of undergraduate classes participated in the 32nd International Mathematics Competition (IMC) held at Blagoevgrad, Bulgaria from 28th July to 3rd August, 2025.

The IMC is an annual event. The first IMC took place in Bulgaria in the year 1994. In the initial years, it took place in different countries like Hungary, UK, Czech Republic, Poland, Macedonia, and Ukraine. For the last 15 years it has been organized by the University College, London, hosted by the American University in Bulgaria and took place in Blagoevgrad, Bulgaria. Though this is an individual event, several universities send their teams and some countries send more than one teams. The Indian students selected on the basis of their performance in MMC participated as the MMC, India team.

The participation of the Indian team may be justly viewed as a significant milestone in the history of math competitions in India, at the undergraduate level. Here, we briefly sketch the journey of reaching this landmark.

India has been participating in the International Mathematical Olympiads (IMO) since 1989. In the last 36 years, the Olympiad competitions have become very prestigious and competitive in India. The impact of Olympiad competitions at regional, national and international levels is stupendous, and the fruits of the efforts put in by students and trainers are overwhelming in terms of the talent pool in mathematics that it has generated in all these years. Motivated by the success of the Olympiad competitions, a group of teachers from Pune, including the present author, thought of launching a math competition for the undergraduate students in India. A proposal for financial assistance submitted to the National Board for Higher Mathematics (NBHM) was readily accepted by the Board. Professor S. G. Dani, then Chairman of NBHM, took keen interest in promoting such a competition. In fact, he named the competition as Madhava Mathematics Competition (MMC) with a view to commemorate the work of a legendary ancient Indian mathematician Madhava, the founder of one of the most influential mathematical traditions in India, the Kerala School of Mathematics, which heralded the pursuit of advanced topics which are now part of the undergraduate curriculum. Madhava and his disciples, over a span of about 250 years developed powerful new mathematics, especially the power series expansions of trigonometric functions, vouching the invention of Calculus in 15th century.

The purpose of launching a competition at the undergraduate level was to inculcate the spirit of problem-solving among students and providing them with an opportunity to experience the excitement of solving math problems in a competitive set up. Initially the competition was organized in two centres, Mumbai and Pune, in 2009. The competition received very good response from the student community as well as from fellow teachers. Mathematicians from several research institutes appreciated the activity and extended their support. As a result, in a few years, the competition blossomed into a major national event, held annually, with more than 20 regions across the country taking part in it. Spread of such an activity requires a lot of active manpower and fortunately, many college teachers came forward and coordinated the event in their regions. Till the year 2023, the competition was organised jointly by S. P. College, Pune and Homi Bhabha Centre for Science Education, TIFR, Mumbai. After that, the organizational centre was shifted to Bhaskaracharya Pratishthana, Pune. The present author has had the privilege and pleasure of being in charge of the activity, as the Chief Coordinator, over the years. The information about the rules of the competition, registration process, syllabus, old question papers, etc. can be found at the website: *https://www.madhavacompetition.in*

In the year 2024, a proposal for participation of Indian team for the international mathematics Competition was sent to NBHM and the members of the Board welcomed the idea and also agreed to provide financial support. It was decided to select top 4 students from MMC-2025 to take part in the International Mathematics Competition for University students. The competition was held from 28th July to 3rd August, 2025 at Blagoevgrad, Bulgaria. The event, hosted by the American University in Bulgaria and organized by University College London, brought together over 430 select undergraduate students from around the world, representing 73 university teams from 52 countries. Some select teams from different countries are listed below:

Sr. No.	Name of Institute	Country
1	Saint-Petersburg State University	Russia
2	Rheinische Friedrich-Wilhelms-Universität, Bonn	Germany
3	Jagiellonian University	Poland
4	Nanyang Technological University	Singapore
5	Tel Aviv University	Israel
6	EPFL-Bernoulli	Switzerland
7	Ecole Normale Supérieure	France
8	University of Bristol	United Kingdom
9	Leiden University	Netherlands
10	Universitate Wien	Austria

The competition took place on two days with 5 questions to be tackled on each day in a span of 5 hours. The topics involved in the IMC are at a more advanced level than the topics for MMC. The IMC papers included in particular the questions on group theory, metric spaces, probability, etc. The old question papers, rules for the participation, results of previous years can be found at the website: *https://www.imc-math.org.uk*
As a part of the preparations for the competition, three training camps were conducted. The details of the camps are as given below:

1. **Online training camp from 22nd May 2025 to 29th May 2025.**
   Resource Persons: V. M. Sholapurkar, S. A. Katre, Mainak Ghosh, Anant Mudgal, Rijul Saini, Rohan Goyal, B. Sury.
   Topics: Advanced problem solving in Linear Algebra, Calculus, Number Theory, Group Theory, Ring Theory.

2. **Training camp at Bhaskaracharya Pratishthana, Pune. 16th June 2025 to 21st June 2025.**
   Resource Persons: Chandrasheel Bhagwat, Sharad Sane, Vinaykumar Aacharya, Geetanjali Phatak, B. Sury, V. M. Sholapurkar, S. A. Katre, Pranjal Jain.
   Topics: Advanced Real Analysis, Combinatorics, Polynomials, Matrix theory, Convex Sets, Topology.

3. **Pre-departure training camp from 26th July 2025 to 27th July 2025 at Bhaskarac-harya Pratishthana, Pune.**
   Resource Persons: Chandrasheel Bhagwat, V. M. Sholapurkar, Vinaykumar Aacharya, Vaidehee Thatte, B. Sury.
   Topics: Advanced Problems in Linear Algebra, Graph Theory, Number Theory, Algebra, Real Analysis, Metric spaces, Set Theory, Permutations, and Polynomial Identities.

The students of the Indian team participated as the Team MMC, India. All the four participants performed very well. Their performance in IMC was as given below:

- **Atmadeep Sengupta** (ISI Kolkata) - **Second Prize** (55 points)
- **Sourodeep Kanjilal** (ISI Kolkata) - **Second Prize** (51 points)
- **Anshuman Agrawal** (ISI Bangalore) - **Second Prize** (46 points)
- **Arnab Sanyal** (ISI Kolkata) - **Third Prize** (42 points)

Sr. No.	Cut-off (in Scores)	Prizes
1	98 - 100	Grand Grand First Prize
2	83 - 93	Grand First Prize
3	56 - 81	First Prize
4	44 - 55	Second Prize
5	34 - 43	Third Prize
6	20 - 33	Honourable Mention
7	0 - 19	Certificate

The points mentioned above are out of 100. The cut-off points for the prizes are indicated in the following table.
Atmadeep missed the first prize by 1 mark and Arnab missed the second prize by 2 marks. Maksim Turevskii of Saint-Petersburg State University, Russia scored 100 marks. In fact, he topped the list for the third consecutive time. 2 students received Grand Grand First Prize, 10 received Grand First Prize and 82 received First Prize.
The team secured 28th position out of 73 participating teams, a notable achievement considering this was the first time for an Indian team to participate in such a competition. The team was accompanied by Prof. B. Sury from ISI, Bangalore and the present author.
International exposure for young budding mathematicians is extremely important in creating an atmosphere conducive to learning higher mathematics. This outstanding debut marks a significant achievement for Bhaskaracharya Pratishthana, Pune and for Indian undergraduate mathematics education. It demonstrates the high calibre of talent emerging from our national platforms and the potential of our students to excel globally in mathematics and allied activities.
As the Chief Coordinator of MMC, the author would like to take this opportunity to heartily acknowledge the support from all the coordinators of MMC for their academic involvement, and to warmly thank NBHM for supporting the activity with generous financial support over the years.

From left to right: 1. Arnab Sanyal, 2. Sourodeep Kanjilal, 3. Prof. V. M. Sholapurkar, 4. Prof. B. Sury, 5. Atmadeep Sengupta, 6. Anshuman Agrawal

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6. Celebrating 100 Years of Prof. R. P. Bambah

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International Conference on Algebra and Number Theory: a report

A two-day International Conference on Algebra and Number Theory (ICANT-2025), was organized by the Department of Mathematics, Panjab University, Chandigarh, during September 30 and October 1, 2025, to commemorate the centenary of **Professor R. P. Bambah**, one of India's most distinguished mathematicians. The conference was initially conceived to celebrate Professor Bambah's 100th birth anniversary - a milestone he narrowly missed by just four months. The conference was supported by the National Board for Higher Mathematics (NBHM), Department of Atomic Energy, Government of India,

The event brought together several leading mathematicians to celebrate the life, scholarship, and enduring legacy of Professor Bambah, a principal architect of the Punjab School of Mathematics.

The inaugural session was graced by Professor Yojna Rawat, Dean University Instruction, Panjab University, as the Chief Guest. In her address, she expressed the emotional significance of the occasion "Though Professor Bambah is no longer with us in person, his spirit, scholarship, and vision continue to illuminate our paths.". She lauded the impact of Prof. Bambah as a scholar as well as an outstanding academic leader, recalling his tenure as Vice-Chancellor of Panjab University (1985-1991), during which the University witnessed notable transformative growth. His honours - including the Padma Bhushan and the Ramanujan Medal - stand as enduring testaments to his greatness.

The present author, Chairperson, Department of Mathematics highlighted Professor Bambah's pioneering contributions to number theory, combinatorics, and geometry, as well as his visionary leadership that made the Department the first Centre for Advanced Study in the Indian university system. The torch Professor Bambah lit decades ago continues to shine brightly, illuminating the paths of countless scholars and inspiring new generations to pursue truth and excellence. His vision led our department with a proud legacy of excellence. Four faculty members have received the Shanti Swarup Bhatnagar Prize and five have served as Presidents of the Indian Mathematical Society.

In her Keynote address by Professor R. J. Hans-Gill, a distinguished student and collaborator of Professor Bambah, detailed the work of Professor Bambah along with his students and grand-students. She emphasized Bambah's leadership focussing especially on the work on Minkowski's Conjecture. For $n = 2$ the conjecture was proved by Minkowski, in 1899. By the early 1970's proofs had been given by various mathematicians for $n = 3, 4$, and 5, but the proofs were complicated, and that for $n = 5$ was not quite rigorous. Bambah, together with A. C. Woods, gave improved proofs in these cases which were simpler and rigorous. He continued to work on related problems and promoted studies on the conjecture. Later, Prof. R. J. Hans-Gill, Prof. V. C. Dumir, Prof. Madhu Raka, and Dr. Leetika Kathuria of the Punjab School extended this work to the cases $n = 7$, 8, and 9.

Bambah and his students also worked on non-homogeneous quadratic forms and published extensively - over 25 papers between 1979 and the late 1990s. Beyond number theory, Bambah contributed to diverse areas such as integer matrices, convex bodies, and coding theory. He authored around 70 research papers from 1946 to 2000.

The conference featured **12 invited talks**, by Prof. M.S. Raghunathan (UM-DAE CEBS, Mumbai), Prof. R. Balasubramanian (IMSc, Chennai), Prof. S. G. Dani (UM-DAE CEBS, Mumbai), Prof. S.D. Adhikari (RKMVERI, West Bengal), Prof. Riddhi Shah (JNU, New Delhi), Prof. Pace Nielsen (Brigham Young University, USA, online), Prof. Ravi Kulkarni (BPRIM, Pune), Prof. Sudhir Ghorpade (IIT Bombay), Prof. K. N. Raghavan (Krea University), Prof. Kapil Paranjape (IISER

Mohali), Prof. D. Surya Ramana (HRI, Prayagraj), Prof. Radha Kessar (University of Manchester, UK, online).
There were two sessions, one on each day, in which young researchers presented their research. Prof. Sudesh Kaur Khanduja, Emeritus Professor, Panjab University, presented the **Prof. R. P. Bambah Memorial Awards** for three best presentations, as adjudged by a committee, to Dr. Gurinder Singh, Dr. Amritpal Singh, and Dr. Jyoti.
A **Remembrance Session** was held on 30 September, honouring Prof. R. P. Bambah's enduring influence on mathematical research in India. Prof. Madhu Raka, Prof. Sudesh Kaur Khanduja, Prof. Radha Kessar (online), Prof. Ajit Iqbal Singh (online), Prof. S.A. Katre, Prof. M.S. Raghunathan, Prof. S.G. Dani, Prof. R. Balasubramanian, and Prof. Kapil Paranjape spoke on the achievements and various laudable features of Prof. Bambah, paying their tributes to him. A video-recorded message was received from Prof. N. Sankaran narrating fond memories of Prof. Bambah, which was presented to the audience. Prof. Bindu Bambah, and Mrs. Charu Bambah, daughters of Prof. Bambah who participated in the event in person, spoke fondly of his influence.
There were over 120 delegates, including several senior faculty members apart from the speakers, from premier institutions such as ISI Delhi, IIT Madras, IISER Mohali, and IIT Roorkee, Bhaskaracharya Pratishthana, Pune, contributing to a rich academic dialogue. The schedule was intensive, running from 9:30 a.m. until late in the evening, but nevertheless, almost all delegates, including the distinguished speakers and senior members, were present throughout the sessions. The enthusiastic participation and engagement by all made the conference a remarkable success.

The Organizing Committee consisting of Surinder Pal Singh Kainth (Convenor), Aarti Khurana (Convenor), Kul Bhushan Agnihotri, and Kathivaran T. was aided by an Advisory Committee with Professors R.J. Hans-Gill, Arun Kumar Grover, Sudesh Kaur Khanduja, Madhu Raka, Gurmeet Kaur Bakshi, and Dinesh Khurana on it, and was enthusiastically assisted by the faculty and the staff from various sections. A team of 20 UG/PG volunteers led by PU Maths Club Coordinator Goyam and Department Representatives Gagan Jindal, Kreetika, and Ranjeet Singh also worked really hard in assisting with organizing this conference.
The conference not only paid tribute to Prof. Bambah's centenary but also reaffirmed the role of the Department of Mathematics, Panjab University, as a hub of mathematical excellence and inspiration for future generations. The Department extended heartfelt gratitude to all distinguished speakers, NBHM, and participants for making the Conference a truly memorable event.

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7. P. P. Divakaran

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P P Divakaran, who passed away recently (at the age of 89), had established himself as a leading expert in the History of Indian mathematics over the last two decades.

Divakran was born in Tellicherry (now Thalassery) and had his schooling there. After a brilliant career at school, he went on to join the Intermediate class of the Madras University at the Government Brenan College in Tellicherry. He passed his Intermediate with flying colours, which enabled him to secure admission to the prestigious BSc (Honours) course in Physics of the University at Presidency College, Madras. After securing the degree (with first class), he got into the Training School of the Bhabha Atomic Research Centre, and the next year(1959), he joined the Tata Institute of Fundamental Research (TIFR) as a Research Scholar. In 1962, he was deputed to the University of Chicago to work for a PhD under the guidance of Professor Dalitz (who had visited TIFR in 1961). When he got back to Bombay (now Mumbai) with his PhD (after a stint in Oxford, where he met and married Odile, a French art historian), he was absorbed into the Physics Faculty at TIFR. During the years at TIFR, he made several significant contributions, mostly to Particle Physics. He retired from TIFR in 1996 and during the next two decades, held visiting appointments at several institutions: Institute of Mathematical Sciences (IMSc) and Chennai Mathematics Institute (CMI) in Chennai, National Centre for Biological Sciences (NCBS) in Bangalore, Inter-University Centre for Astronomy and Astrophysics (IUCAA) in Pune, Harish-Chandra Research Institute (HRI) in Prayagraj. He was awarded a Homi Bhabha Fellowship specifically to work on the History of Mathematics, and IUCAA hosted him during that fellowship.

A chance meeting he had with K. V. Sarma, a historian, triggered in him an interest in the history of Indian mathematics, especially in the mathematics that emanated from Kerala during the 14th to 17th century. If my memory serves me right, I was with him at this meeting with K. V. Sarma. So, it was somewhat late in life that he developed this interest, but he showed a youthful passion in pursuing it. He continued thinking of problems in Physics, and as late as early this year, he spoke to me excitedly about some ideas he had connected to Relativity.

I joined TIFR in 1960 as a Research Scholar in Mathematics and got to know him within a few weeks of my joining there. Our first meeting turned out to be somewhat amusing. When I learnt that he was a Malayalee, I asked him where in Kerala he was from. He said, "Oh, it is a small town, you wouldn't have heard of. It is called Tellicherry". As it turned out, I knew of Tellicherry and had in fact spent a few months there as a 6-year-old! My maternal grandfather, P. R. Krishnaswamy, was the principal of the Government Brenan College in Tellicherry during 1946-49. Divakaran himself was a student of the College, but that was after my grandfather's time; an elder sister of his studied at the college during my grandfather's tenure, and so he had heard my grandfather's name.

Divakaran's was an incisive mind, and his intellectual interests were wide and well beyond his primary preoccupation, Theoretical Physics. He was a brilliant conversationalist with a lively sense of humour that made his company most enjoyable. He was a bon vivant with a refined taste in anything that interested him. He did not suffer fools gladly. He hung out with mathematics students more often than other Physics students. He counted among his close friends two mathematics students, Ramanan and Raghavan Narsimhan. I was the third one to get close to him. Once during our hostel days, Fritz Staal, a Dutch student of Philosophy working in Madras University, specialising in Vedic Rituals (who became a well-known Philosopher later) came visiting Divakaran. Divakaran had known him in his Madras days. I ran into Divakaran, Stahl, and Ramanan engaged in an animated discussion about Vedic rituals. I knew that Divakaran had a wide range of interests, but was surprised to find that it included Vedic studies. Ramanan came from a Tamil Brahmin family that was very much into Vedic religious practices.

Divakaran's work in Physics tended to be mathematical, and he was one of the few physicists at TIFR who had serious interactions with mathematicians. I happen to be one of those mathematicians. I was greatly impressed with his grasp of mathematical concepts that he put to use in his work. Our collaboration resulted in my writing an Appendix to a paper of his in Communications in Physics. While I could see that he grasped the mathematical concepts I talked about to him, I am afraid that I had, even at that time, at best, a hazy idea of the Physics he applied them to in that paper; now, after three decades, I can recall nothing about it. He had much more intensive interactions with my colleague M. S. Narasimhan.

Divakaran was pretty good at tennis and was, along with Obaid Siddiqi, one of the two best Tennis players in TIFR. I was not a particularly good player, but Divakaran was nice enough to me to play with me often. His proficiency in tennis, perhaps, enabled him to become a member of Bombay Gymkhana, admission to which was far from easy. Divakaran's wife was an Art Historian, and he travelled a great deal with her on her professional trips to visit sites of ancient sculptures all over India. His travels with Odile and the membership of Bombay Gymkana led to his acquiring a wide circle of friends well beyond academia: the painter Jehangir Sabavala and his wife were, for example, family friends of his.

After he retired from TIFR in 1996, he was much sought after as a Visiting Professor, thanks to his wide scholarship. In about a decade or so following his retirement from TIFR, he got interested seriously in the History of Mathematics and set about researching in the subject. He became a well-recognised scholar-researcher in the area, bringing new insights to it.

A weekly seminar on *Topics in the History of Indian and Western Mathematics* was held at the Chennai Mathematical Institute in January-February 2008, an endeavour at the initiative of David Mumford who was then visiting the Institute, with the enthusiastic support of C. S. Seshadri, the then Director of the Institute, and the pro-active participation of Divakaran. There were talks by several invited speakers on a wide range of topics in ancient Indian mathematics and a volume consisting of papers associated with the presentations was published, which includes, in particular, a voluminous contribution by Divakaran (cf. [2]), his first paper in the history of mathematics, on the Kerala mathematical treatise Yuktibhasha. Many more were to follow on the theme, setting him apart as a unique expert on the mathematics of the Kerala school, which he preferred to call the Nila school, in the context of its having flourished on the banks of river Nila, and its significance in the development of calculus.

His scholarship on Indian mathematics did not remain limited to Kerala mathematics however. Interestingly, during his IUCAA years he collaborated with a young student of Vedic Sanskrit, Bhagyashri Bavare, in analysing the numbers appearing in the Rigveda, producing an incisive analysis of how they were composed and its significance in the development of the decimal system of number representation (cf. [1]). Another of his important papers [3] brings out the unique features of Indian mathematics and how it differs from the mathematics of other cultures.

With the background of the wide range of papers, writing a book was in order, and in 2018, he published one titled "The Mathematics of India", (cf. [4]) which is an outstanding work encompassing the history of mathematics in India from the Indus Valley Civilisation days to the beginnings of British rule. It is less comprehensive than Kim Plofker's work, one of the standard references in the area, but it has many new insights, insights that are perhaps the result of Divakaran's greater familiarity with Indian culture and his knowledge of his mother tongue, Malayalam (the language of the important work Yuktibhasha), as well as a deeper knowledge of mathematics itself. The book may well be considered a worthy successor of the pioneering work of Datta and Singh; the same may be said of Plofker's book, but Divakaran's book is better able to trace the continuities in the Indian mathematical tradition. The chapters of the book devoted to Madhava and his school are a veritable treat. The book received adulatory reviews from, among others, eminent mathematicians M. S. Narasimhan (cf. [5]) and Avinash Sathaye (cf. [6]). Divakaran's command of English was indeed superb.

A satellite conference on History of Mathematics was held following the International Congress of Mathematicians 2010, at the Kerala School of Mathematics, Calicut. Divakaran served as a member of the Organizing Committee. Among other things in that role he relished guiding a tour of interested delegates, which included many foreigners, to the neighbourhood where the mathematics of the Kerala school of Madhava flourished, including a visit to a family of descendants of Parameswara, one of the renowned exponents from the School.

Divakaran held strong views on many issues. While he had great admiration for the mathematics of our ancestors, he rubbished the exaggerated chauvinistic claims about India's mathematics made by some of our compatriots. He was a liberal in politics with a strong commitment to human rights and was much distressed by the developments that curtailed them in recent years. Even as he devoted much time to researching the history of mathematics, he continued to think about problems in Physics.

His work in Physics as well as in History of Mathematics won him the Kairali Global Lifetime Achievement Award (2022-23).

In his passing away, we have lost an outstanding scholar-researcher of the history of mathematics, who was very active in the field despite his advanced age. Personally, I have lost a good friend of many decades.

Acknowledgement: I am grateful to S. G. Dani for some important inputs used in the text above.

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Compact Course on Complex Analysis, MSU, Vadodara

Participants

8. Problem Corner

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In the last issue we presented two challenging problems from Number Theory and Combinatorics. A solution of Number Theory from the problem proposer is here. Still no solutions are received for some of the problems of past issues. Hopefully we may get solutions of some of them in forthcoming issues.
In this issue, we pose two problems one from Algebra by Dr. Vinaykumar Acharya and one from Number Theory by Amarnath Murthy and K B Subramaniam for our readers. Readers are invited to email their solutions to Dr. Udayan Prajapati (*ganit.spardha@gmail.com*), Coordinator, Problem Corner before 10th December, 2025. Most innovative solution will be published in the subsequent issue of bulletin.
The first Problem posed in the last issue:
Let a and b be distinct positive integers such $3^a + 2$ is divisible by $3^b + 2$. Prove that $a > b^2$.
Solution by Dr. Vinaykumar Acharya. Obviously we have $a > b$. Let $a = bq + r$, where $q > 0$, $0 \le r < b$. Then

$$-2 \equiv 3^a \equiv 3^{bq+r} \equiv (-2)^q(3^r) \pmod{3^b+2}. \qquad (1)$$

So $3^b + 2$ divides $A = (-2)^q 3^r + 2$ and it follows that

$$|(-2)^q 3^r + 2| \ge 3^b + 2 \text{ or } (-2)^q 3^r + 2 = 0.$$

We make case distinction:

1. $(-2)^q 3^r + 2 = 0$. Then $q - 1$ and $r = 0$ which implies that $a = b$, a contradiction.
2. q is even. Then $A = 2^q 3^r + 2 = (3^b + 2)k$ for some positive integer k. Consider both sides of the last equation modulo 3^r. Since $b > r$:

   $$2 \equiv 2^q 3^r + 2 = (3^b + 2)k \equiv 2k (\text{mod } 3^r),$$

   so it follows that $3^r | (k - 1)$. If $k = 1$ then $2^q 3^r = 3^b$, a contradiction. So $k \ge 3^r + 1$, and therefore:

   $$A = 2^q 3^r + 2 = (3^b + 2)k \ge (3^b + 2)(3^r + 1) > 3^b 3^r + 2.$$

   It follows that

   $$2^q 3^r > 3^b 3^r, \text{ i.e. } 2^q > 3^b, \text{ which implies } 3^{b^2} < 2^{bq} < 3^{bq} \le 3^{bq+r} = 3^a.$$

   Consequently $a > b^2$.
3. q is odd. Then $2^q 3^r - 2 = (3^b + 2)k$ for some positive integer k.
   Considering both sides of the last equation modulo 3^r, and since $b > r$, we get: $k + 1$ is divisible by 3^r and therefore $k \ge 3^r - 1$.
   If $r = 0$, as $k \ge 1$, $2^q - 2 \ge 3^b + 2$, so $2^{q+1} > 3^b$.
   If $r > 0$, $2^q 3^r - 2 = (3^b + 2)k \ge (3^b + 2)(3^r - 1)$, and therefore
   $2^q 3^r > (3^b + 2)(3^r - 1) > 3^b(3^r - 1) > 3^b \frac{3^r}{2}$, so $2^{q+1} > 3^b$.
   But for $q > 1$ we have $2^{q+1} < 3^q$, which combined with the above inequality, implies that

   $$3^{b^2} < 2^{(q+1)b} < 3^{qb} \le 3^a,$$

   q.e.d. Finally, if $q = 1$ then $2^q 3^r - 2 = (3^b + 2)k$ and consequently

   $$2 \cdot 3^r - 2 \ge 3^b + 2 \ge 3^{r+1} + 2 > 2 \cdot 3^r - 2,$$

   a contradiction.

Problems for this issue

Problem 1: Let $p(x)$ be the polynomial left as the remainder when $x^{2025} - 1$ is divided by $x^6 + 1$. What is the remainder, when $p(x)$ is divided by $x - 3$?

Problem 2: Prove that for a given prime p there exists a corresponding positive integer n_0 such that for all integers n greater than n_0, there exists some positive integer k (depending on n) such that $n - kp$ is a prime.

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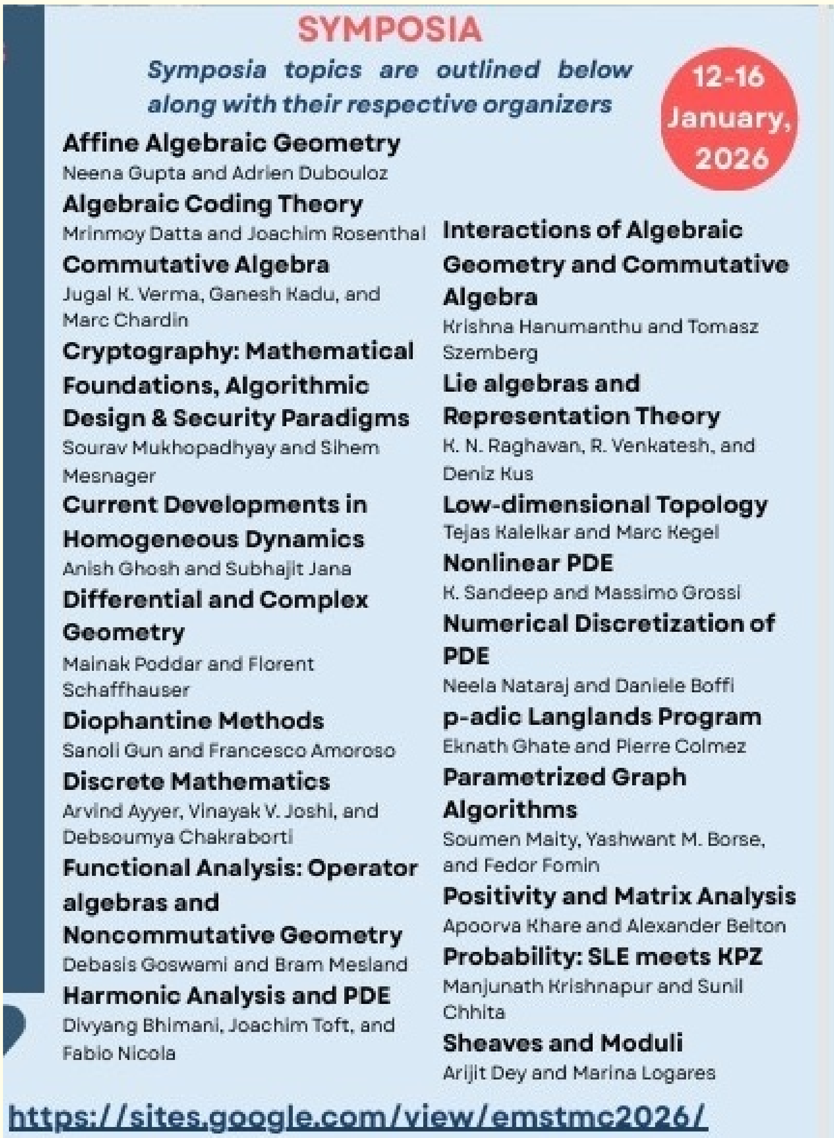
Indo-European Conference in Mathematics



9. International Calendar of Mathematics Events

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January 2026

- January 5-9, 2026, Extremal Black Holes and the Third Law of Black Hole Thermodynamics, ICERM/ Brown University, Providence, Rhode Island.
icerm.brown.edu/program/topical_workshop/tw-26-ebh
- January 11-14, 2026, ACM-SIAM Symposium on Discrete Algorithms (SODA26) Co-Located with SIAM Symposium on Algorithm Engineering and Experiments (ALENEX26) and SIAM Symposium on Simplicity in Algorithms (SOSA26), Hyatt Regency Vancouver, Vancouver, Canada. *www.siam.org/conferences-events/siam-conferences/soda26/*
- January 12-16, 2026, Indo-European Mathematics Conference on Mathematics, jointly organised by the European Mathematical Society and The (Indian) Mathematics Consortium, S. P. Pune Univ., and IISER, Pune. *https://sites.google.com/view/emstmc2026/*
- January 12-16, 2026, AIM Workshop: Formal Scientific Modeling: A Case Study in Global Health, American Institute of Mathematics, Pasadena, California.
aimath.org/workshops/upcoming/formalmodel/
- January 12-16, 2026, Nonparametric Bayesian Inference - Computational Issues ICERM/ Brown University, Providence, Rhode Island.
icerm.brown.edu/program/topical_workshop/tw-26-bnp
- January 12-16, 2026, Nonparametric Bayesian Inference - Computational Issues, Institute for Computational and Experimental Research in Mathematics, Brown University, Providence, RI 2903. *icerm.brown.edu/program/topical_workshop/tw-26-bnp*
- January 15-17, 2026, A conference on "Recent Developments in Applied and Computational Mathematics (RDACM 2026)" in honour of Prof. Raju K. George, Department of Mathematics, Indian Institute of Space Science and Technology, Trivandrum, India.
- January 20 - May 22, 2026, Geometry and Dynamics for Discrete Subgroups of Higher Rank Lie Groups, SL Math 17, Gauss Way Berkeley CA 94720. *www.slmath.org/programs/365*
- January 20 - May 22, 2026, Topological and Geometric Structures in Low Dimensions, SL Math 17, Gauss Way Berkeley CA 94720. *www.slmath.org/programs/368*
- January 21-23, 2026, Connections Workshop: Topological and Geometric Structures in Low Dimensions & Geometry and Dynamics for Discrete Subgroups of Higher Rank Lie Groups SL Math, 17 Gauss Way, Berkeley CA. *www.slmath.org/workshops/1124*
- January 26-30, 2026, Introductory Workshop: Topological and Geometric Structures in Low Dimensions & Geometry and Dynamics for Discrete Subgroups of Higher Rank Lie Groups SLMath, 17 Gauss Way, Berkeley CA. *www.slmath.org/workshops/1125*

February 2026

- February 2-6, 2026, AIM Workshop: Time-Dependent Bernoulli-Type Free Boundary Problems, American Institute of Mathematics, Pasadena, California.
aimath.org/workshops/upcoming/bernoullievol/
- February 10-12, 2026, International Conference on Advanced Scientific Computing & Machine Learning (ASCML 2026), BITS Pilani K K Birla Goa Campus, Goa, India.
www.bits-pilani.ac.in/goa/ascml2026/

March 2026

- March 7-8, 2026, 2026 Spring Western Sectional Meeting, Boise State University, Boise, ID.
www.ams.org/meetings/sectional/2327_program.html
- March 23-27, 2026, Recent Progress in Topological and Geometric Structures in Low Dimensions, S L Math, 17 Gauss Way, Berkeley, CA. *www.slmath.org/workshops/1141*

- March 28-29, 2026, 2026 Spring Southeastern Sectional Meeting, Armstrong Campus Of Georgia Southern University, Savannah, Savannah, GA. *www.ams.org/meetings/sectional/2329_program.html*
- March 28-29, 2026, 2026 Spring Eastern Sectional Meeting, Boston College, Boston, MA. *www.ams.org/meetings/sectional/2331_program.html*

April 2026

- April 13-17, 2026, AIM Workshop: Combinatorial Coding Theory, American Institute of Mathematics, Pasadena, California. *aimath.org/workshops/upcoming/combincoding/*
- April 13-24, 2026, GAP XX - Kyoto: Moduli Spaces in Various Flavors of Geometry, RIMS, Kyoto University, Kyoto, Japan. *sites.google.com/view/gap2026/home*
- April 18-19, 2026, 2026 Spring Central Sectional, North Dakota State University, Fargo, ND. *www.ams.org/meetings/sectional/2323_program.html*
- April 20-23, 2026, Valentia Geometrica 2026: A Workshop on Geometric Analysis, Universitat De Valencia, Valencia, Spain. *sites.google.com/view/valentiageometrica26/valentia-geometrica*
- April 27 - May 1, 2026, Nonlinear Waves: From Theoretical and Computational Advances to Experimental Observations, ICERM/Brown University, Providence, Rhode Island. *icerm.brown.edu/program/topical_workshop/tw-26-tcaeo*

May 2026

- May 4-15, 2026, Séminaire De Mathématiques Supérieures 2026: Universal Statistics In Number Theory, Montréal, Canada. *www.slmath.org/summer-schools/1149*
- May 11-15, 2026, 2026 NSF-CBMS conference on Strong Matrix Properties And The Inverse Eigenvalue Problem, Eastern Michigan University. *sites.google.com/emich.edu/cbms*
- May 11-15, 2026, Techniques And Tools For The Formalization Of Analysis ICERM /Brown University, Providence, Rhode Island. *icerm.brown.edu/program/topical workshop/tw-26-ttfa*
- May 28-31, 2026, Fourth International Conference On Applications Of Mathematics To Nonlinear Sciences (AMNS-2026), Dhulikhel/Kathamndu, Nepal. *anmaweb.org/AMNS-2026/*

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INDO-EUROPEAN CONFERENCE IN MATHEMATICS

Jointly organized by

European Mathematical Society
and
The (Indian) Mathematics Consortium

Jointly hosted and co-organized by

Savitribai Phule Pune University (SPPU)
and
Indian Institute of Science Education and Research (IISER), Pune

Plenary Speakers

Adi Adimurthi
TIFR CAM, Bengaluru, India
Nathanaël Berestycki
University of Vienna, Austria
Manjul Bhargava
Princeton University, USA
Arup Bose
ISI Kolkata, India
Pavel Etingof
MIT, Cambridge, USA
Neena Gupta
ISI Kolkata, India
Chandrashekhar Khare
UCLA, Los Angeles, USA
Ariane Mézard
Sorbonne Université, France
Siddharth Mishra
ETH Zürich, Switzerland
Nitin Saxena
IIT Kanpur, India
Claire Voisin
Collège de France, Paris

12-16 January, 2026

Venue:
**SPPU and IISER Pune
Pune, India**

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Meena Mahajan
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The conference features 11 plenary talks, 20 symposia on different areas of mathematics, contributed talks, and poster presentations. For details, visit:
https://sites.google.com/view/emstmc2026/

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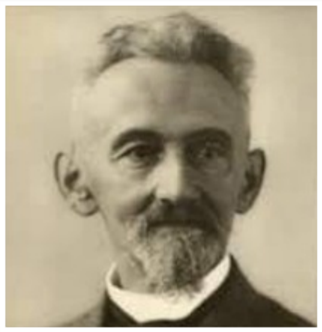
Last date for Submission of Contributed Talk or Poster: **31st October 2025**

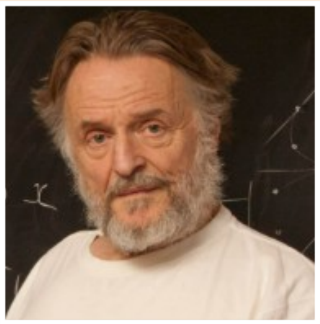
Wolfgang M Schmidt (03 Oct. 1933 -)

An Austrian mathematician. Made profound and far-reaching contributions to number theory. Published over 200 research articles, containing spectacular results and breakthroughs in Diophantine approximation, geometry of numbers, uniform distribution and normality. Known for Schmidt Subspace Theorem; the axiomatic formulation of Schmidt Games, and the unifying aspect of Regular Systems. Recipient of the AMS Cole Prize, the Humboldt Prize.



Felix Hausdorff (08 Nov. 1868 - 26 Jan. 1942)

A German mathematician. One of the founders of modern topology and who Contributed significantly to the theory of ordered sets (introduced partially ordered sets), the study of cardinal and ordinal numbers, measure theory, function theory, and functional analysis. Formalized the concept of a topological space using neighborhood axioms. Known for Hausdorff Spaces, Hausdorff measure and Hausdorff dimension useful in the study of fractals.



John Horton Conway (26 Dec. 1937 - 11 April 2020)

An English mathematician. Invented the Game of Life, one of the early examples of a cellular automaton. Contributed to combinatorial game theory. Known for Conway polyhedron notation, the Conway criterion to identify many prototiles that tile the plane, system of notation for tabulating knots, the Conway groups. Gave a counterexample to the converse of the intermediate value theorem. Conway's conjecture on thrackle, is still open.

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