The Mathematics Consortium

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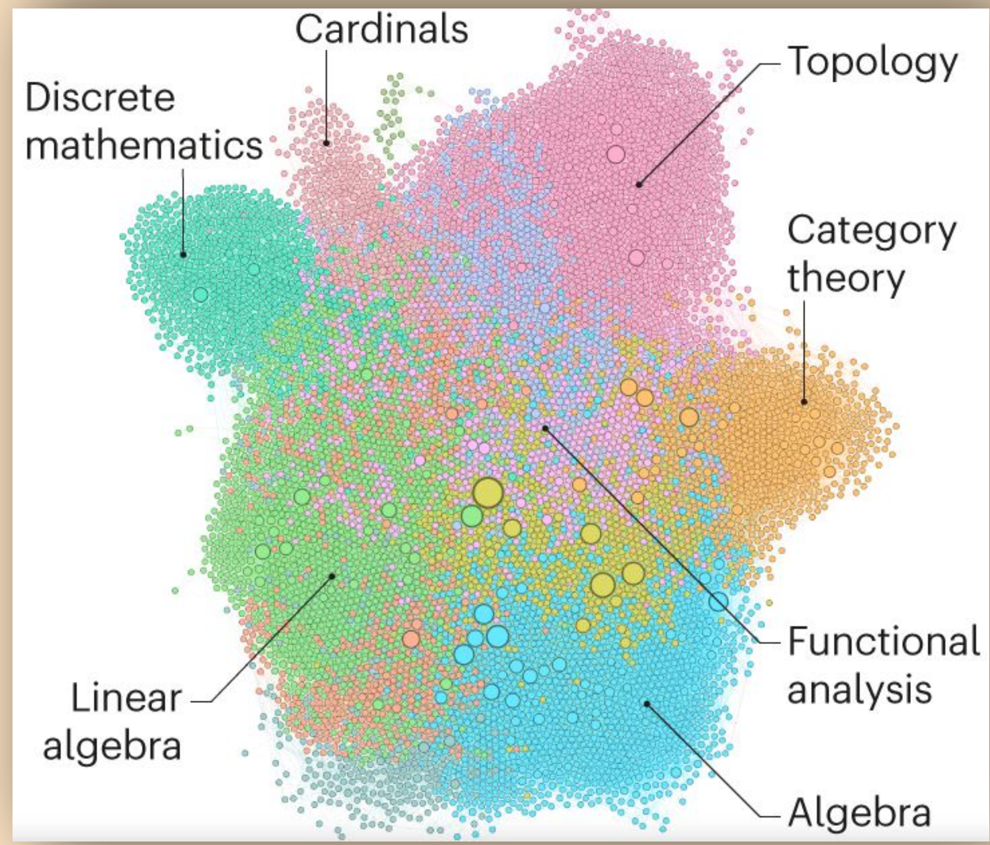
A proof-assistant software Lean



Typical output of Lean in the form of a complex Network

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Contents

About the Cover-page: *In the proof-assistant package Lean, users enter mathematical statements based on simpler statements and concepts that are already in the Lean library. The output, seen here in the case of Scholze and Clausen's key result, is a complex network. The statements have been colour-coded and grouped by subfield of mathematics.*
Credit: Patrick Massot
Source: https://www.nature.com/articles/d41586-021-01627-2

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From the Editors' Desk

There have been several attempts in history to reach a unified theory of mathematics. Many renowned mathematicians have held that the whole subject should be put on a common platform with a self-consistent body of definitions, axioms, theorems, examples, and so on which may be adopted to all subfields of Mathematics.

A major milestone in these efforts, was development of Category theory (regarded as a unified theory of mathematics), by American mathematicians Saunders Mac Lane and Samuel Eilenberg in 1945. A key theme from the "categorical" point of view is that mathematics requires not only certain kinds of objects but also mappings between them that preserve their structure.

In 2019, Peter Scholze, a number theorist, based at University of Bonn, Germany, proposed a program for Condensed mathematics, a project to unify various mathematical subfields, including topology, geometry, and number theory, in collaboration with Dustin Clausen from the University of Copenhagen. In 2020 Scholze authored an involved proof that was critical to the theory. He asked other mathematicians led by Johan Commelin to help him verify its correctness. Over a 6-month period the group verified the proof using the proof assistant software Lean.

Mathematicians have long used computers to do numerical calculations or manipulate complex formulas. In the 1970s, they have solved the most famous Four colour problem by making computers do massive amounts of repetitive work. However, Proof-assistant software handles an abstract concept at the cutting edge of research, revealing a bigger role for software in mathematics. These developments in Mathematics and computing techniques will certainly influence the research methodologies and the graduate courses in Mathematics in near future.

This is included as one of the news items in the Article 6 in which Dr. D. V. Shah gives an account of the Important events which occurred in the Mathematics world during last three months. Also, in Article 2, Prof. S. A. Katre shows that various Interpolation formulae can be easily derived using a Polynomial version of Chinese Reminder theorem, sort of unifying ideas in Number theory and Numerical Analysis.

In Article 1, Prof. Steven Dale Cutkosky from University of Missouri, USA discusses Abhyankar's fundamental contributions to the field of resolution of singularities. This article is based on the talk given by him at Bhaskaracharya Pratishthan in July 2021. In Article 3, Prof. G. V. V. Hemasunder gives historical perspective of the Uniformisation theorem and related developments leading to its complete proof. He focuses on Koebe's General Uniformisation theorem which is more general than the Uniformisation theorem.

In Article 4, Prof. Rajat Tandon narrates the interesting story of ICM 2010 held at Hyderabad and decisive role played by Prof. M. S. Raghunathanin in organizing the ICM. A review by Prof. P. Shunmugraj of the book by Sudhir Ghorpade and Balmohan Limaye, A Course in Calculus and Real Analysis, 2nd Ed., Springer, 2018, is included in Article 5.

In Problem Corner, Dr. Udayan Prajapati presents a solution to the problem posed in the previous issue given by Rosna Paul, a research scholar at Technical University, Graz, Austria, and poses a problem from Combinatorics for our readers. Dr. Ramesh Kasilingam gives a calendar of Academic events, planned during November, 2021 to January 2022. Prof. Sudhir Ghorpade gives update on TMC-Distinguished Lecture Series, Prof. T. R. Ramdas provides information about IMU publications and Prof. Katre gives the update on TMC activities.

We congratulate our Indian team for their commendable performance at IMO 2021. We also pay our tributes to a well-known Indian mathematician Prof. Inder Bir Singh Passi, a noted group-theorist, who passed away on 2nd October, 2021 at the age of 82.

We are very happy to bring out the second issue of Volume 3 in October 2021. We thank all the authors, all the editors, our designers Mrs. Prajkta Holkar and Dr. R. D. Holkar and all those who have directly or indirectly helped us in bringing out this issue on time.

Chief Editor, TMC bulletin.

1. Abhyankar's Fundamental Work on Resolution of Singularities

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Abstract: We discuss Abhyankar's fundamental contributions to the field of resolution of singularities. We further review some key points of what is now known and not known about resolution of singularities in positive characteristic. This is an extended version of the talk given by the author at the virtual conference "A Symposium in Mathematics on the Birthday of late Professor Shreeram S. Abhyankar", July 22, 2021, held by Bhaskaracharya Pratishthana.

1.1 Algebraic geometry and valuations

We first review some notations and facts about valuation theory in algebraic geometry, as developed by Zariski.

The maximal ideal of a local ring R will be denoted by m_R. An algebraic function field K over a field k is a finitely generated field extension of k. The dimension d of K is the transcendence degree of K over k. There is then a finite extension $k(x_1,\ldots,x_d) \rightarrow K$ where $k(x_1,\ldots,x_d)$ is a rational function field in d variables.

An algebraic local R of K is a subring of K which is a localization of a finitely generated k-algebra and has K as its quotient field.

An algebraic variety X over a field k can be thought of as a collection of algebraic local rings of the algebraic function field $K = k(X)$. A projective variety is the set of local rings of a closed subvariety of projective space over k.

Let K be an algebraic function field over k. A valuation ν of K is a surjective map $\nu : K^{\times} \rightarrow \Gamma$ where Γ is a totally ordered abelian group such that

1. $\nu(fg) = \nu(f) + \nu(g)$ for $f, g \in K^{\times}$
2. $\nu(f+g) \geq min\{\nu(f), \nu(g)\}$
3. $\nu(c) = 0$ if $c \in k^{\times}$.

The first two conditions are that ν is a valuation of the field K. The third condition is imposed by the extra structure that K is an algebraic function field over k. The valuation ring of ν is $V_\nu = \{f \in K \mid \nu(f) \geq 0\}$ with maximal ideal $m_\nu = \{f \in K \mid \nu(f) > 0\}$. We are following the convention that $\nu(0) = \infty$.

Let R be an algebraic local ring of a function field K and ν be a valuation of K. We say that R is dominated by ν if $R \subset V_\nu$ and $m_\nu \cap R = m_R$. Given a projective variety X and a valuation ν of the function field $K = k(X)$ there exists a unique point q of X whose local ring is dominated by ν. We say that q is the center of ν on X.

Let R be an algebraic local ring of a function field K, I be a nonzero ideal in R and ν be a valuation of K such that R is dominated by ν. There exists $f \in I$ such that $\nu(f) = min\{\nu(g) \mid g \in I\}$. We have an inclusion of algebraic local rings $R \rightarrow R_1$ which are dominated by ν where

$$R_1 = \left(R\left[\frac{I}{f}\right]\right)_{m_\nu \cap R[\frac{I}{f}]}$$

The extension $R \rightarrow R_1$ is called a blow up along ν. A particular example is the normalization. Let $\overline{R}$ be the integral closure of R in K. Then $R \rightarrow \overline{R}_{m_\nu \cap \overline{R}}$ is a blow up along ν.

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Suppose that X and Y are projective varieties. An algebraic mapping $\varphi : Y \to X$ is said to be birational if φ is an isomorphism over an open subset of X. This is equivalent to the condition that φ induces an isomorphism of function fields. We can thus identify the function fields $k(X)$ and $k(Y)$, which we will refer to as K. In our interpretation of geometry as a set of local rings in a function field K, we can view X and Y as two sets of local rings of K. The map φ takes a point $q_1 \in X$ if and only if $R \subset R_1$ and $m_{R_1} \cap R = m_R$, where R is the local ring of q on X and R_1 is the local of q_1 on Y.

Any birational mapping $\varphi : Y \to X$ of projective varieties is locally the blow up on ideal. Let ν be a valuation of the function field K of X and Y, q_1 be the center of ν on Y and q be the center of ν on X. Let R be the local ring of q on X and R_1 be the local ring of q_1 on Y. Then $R \to R_1$ is a blow up along ν.

We have thus found that fixing a valuation ν gives us a correspondence from algebraic geometry to local algebra, and if we consider all valuations of the function field, we obtain a 1-1 correspondence. This is a philosophy of algebraic geometry which was first introduced for curves, and later developed in all dimensions by Zariski.

1.2 Local uniformization and resolution of singularities

A local ring R is a regular local ring if $dim_{R/m_R} m_R/m_R^2 = d$ where d is the dimension of the ring. A variety X is nonsingular if all of its local rings are regular local rings.

Suppose that R is an algebraic local ring with $k = R/m_R$. If k is algebraically closed, then R has a representation $R = (k[x_1, \ldots, x_n]/P)_m$ where $P = (f_1, \ldots, f_r)$ is a prime ideal in the polynomial ring $k[x_1, \ldots, x_n]$ and $m = (x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal, where $\bar{a} = (a_1, \ldots, a_n) \in k^n$. Let d be the dimension of R. Then we have the Jacobian criterion

$$R \text{ is a regular local ring if and only if rank} \left(\frac{\partial f_i}{\partial x_j}(\bar{a}) \right) = n - d.$$

If we remove the assumption that k is algebraically closed, and allow $k \to R/m_R$ to be a finite extension, then the $\Leftarrow$ direction of the Jacobian criterion is true but the $\Rightarrow$ direction can fail if there is inseparability in the extension $k \to R/m_R$.

A resolution of singularities of a projective variety X is a birational morphism $\varphi : Y \to X$ where Y is a nonsingular projective variety. The local form of a resolution of singularities was called local uniformization by Zariski. Let K be an algebraic function field. Local uniformization holds for K if whenever ν is a valuation of K, there exists an algebraic regular local ring of K which is dominated by ν.

Given a function field K and a projective variety X with function field K, if X has a resolution of singularities, then local uniformization holds for K. This can be seen quite simply. Let $X_1 \to X$ be a resolution of singularities. Since X_1 is projective, every valuation of K dominates a local ring of X_1, and since X_1 is nonsingular, all local rings of X_1 are regular.

1.3 Resolution of singularities and local uniformization in characteristic zero

A projective curve X over any field can be resolved just by taking its normalization, so the problem of resolution only becomes interesting in dimension ≥ 2. Walker [13] proved resolution of singularities of complex surfaces in 1938.

Zariski proved the following fundamental theorem about local uniformization, in a paper which appeared in 1940.

Theorem 1. *(Zariski [15]) Local uniformization holds for K whenever K is a characteristic zero algebraic function field.*

We showed above that if resolution of singularities holds for a projective variety X, then local uniformization holds for the algebraic function field of X. Conversely, if X is a projective variety of dimension less than or equal to 3 and local uniformization holds for the function field of X, then resolution of singularities holds for X. Zariski showed this in characteristic zero [14], [16] and Abhyankar later extended this to all characteristics 3. This converse statement is still unknown in dimension larger than three.

Corollary 2. *(Zariski [14], [16]) Resolution of singularities is true for characteristic zero algebraic varieties in dimension* ≤ 3*.*

In 1964, Hironaka, who had been Zariski's student, proved resolution of singularities in characteristic zero and in all dimensions.

Theorem 3. *(Hironaka [8]) Resolution of singularities is true for characteristic zero algebraic varieties of arbitrary dimension.*

Hironaka's proof does not use valuation theory. The starting point of this extraordinary proof is the use of the Tschirnhaus transformation which gives a "hypersurface of maximal contact" for the singularity. This amazing idea was found by Abhyankar. Unfortunately, it only works in characteristic zero. Indeed, Narasimhan [12], in his PhD thesis with Abhyankar, gives an example of a positive characteristic singularity which does not have a hypersurface of maximal contact. This failure of the Tschirnhaus transformation in positive characteristic is the only part of Hironaka's proof which does not extend to positive characteristic.

1.4 What about characteristic p?

From now on, we assume that the ground field k is algebraically closed, unless stated otherwise, and assume that varieties are projective.

In his PhD thesis with Zariski, which appeared in [2] in 1956, Abhyankar proved local uniformization and resolution of singularities for surfaces, and later, culminating in his 1966 book [3], Abhyankar proved local uniformization and resolution of singularities for 3-folds over fields of characteristic > 5.

Theorem 4. *(Abhyankar [2], [3]) Local uniformization is true in two dimensional algebraic function fields of arbitrary characteristic and in three dimensional algebraic function fields over algebraically closed fields of characteristic* $p > 5$*.*

Corollary 5. *(Abhyankar [2], [3]) Resolution of singularities is true for algebraic varieties of dimension two and for algebraic varieties of dimension three and in characteristic > 5.*

Another important theorem in resolution is embedded resolution of singularities. Let X be a nonsingular variety of dimension $d+1$ and Z be a union of subvarieties of X of dimension d. Then locally in X, Z is given by the vanishing of a single equation. An embedded resolution of X is a birational mapping $\varphi : X_1 \to X$ such that X_1 is nonsingular, and $\varphi^{-1}(Z)$ has simple normal crossings (SNCs). That is, at every point q of $\varphi^{-1}(Z)$, there exist regular parameters $x_1, \ldots, x_d + 1$ in the local ring $\mathcal{O}_{X_1,q}$ of q in X_1 and $1 \leq r \leq d+1$ such that $x_1 x_2 \ldots x_r = 0$ is a local equation for $\varphi^{-1}(Z)$.

Abhyankar [3] proved embedded resolution of singularities in dimension 3. It is an essential ingredient of his proof of resolution of singularities of 3-folds. The proof in [3] draws on a series of his previous papers.

Theorem 6. *(Abhyankar [3]) Embedded resolution of singularities is true in dimension 3 (embedded resolution of* Z *is true if* Z *is a union of two dimensional subvarieties of a nonsingular 3-fold* X*).*

A simpler proof of Abhyankar's results on resolution in dimension three can be found in [7].

More recently, resolution of singularities has been proven in dimension ≤ 3 for the most general cases of schemes for which resolution could possibly be true. Resolution of singularities of reduced excellent schemes of dimension two has been proven by Lipman [11] and embedded resolution of reduced excellent schemes of dimension two has been proven by Cossart, Jannsen and Saito [5]. Resolution of singularities of reduced excellent schemes of dimension three has been proven by Cossart and Piltant [6] in 2019. Restricting to varieties over a field, these results tell us that resolution of singularities is true for two dimensional algebraic varieties over an arbitrary field and that resolution of singularities is true for three dimensional algebraic varieties over an arbitrary field; that is, Abhyankar's theorems on resolution are true with no restriction on the ground field.

Both resolution of singularities and local uniformization are open problems in dimension ≥ 4 in all positive characteristics as of the time of this writing. Embedded resolution of singularities is open in dimension ≥ 4 in all positive characteristics. Thus although we can resolve singularities of 3-folds, we cannot construct an embedded resolution of a 3-fold inside a nonsingular 4-fold.

There is a very important resolution theorem by de Jong [9] in 1996, which is true quite generally. He proves resolution of singularities after taking a suitable finite extension; that is after a non birational extension in a finite extension of function fields. Let X be an algebraic variety. An alteration of X is a map $\varphi : Y \to X$ where Y is nonsingular and where φ has a factorization $Y \to Z \to X$, where $Z \to X$ is a normalization of X inafinite field extension of the function field $k(X)$ of X and $Y \to Z$ is birational.

Theorem 7. *(de Jong [9]) Let X be an algebraic variety over an arbitrary field k. Then there exists an alteration $\varphi : Y \to X$. If the ground field k of X is perfect, then the function field $k(Y)$ is a separable field extension of $k(X)$.*

1.5 Some methods of proof

Zariski suggested while Abhyankar was a PhD student that he try to extend a classical method of Jung to prove resolution of singularities in positive characteristic. Abhyankar proved the following theorem, which appeared in 1956.

Theorem 8. *(Abhyankar-Jung theorem) (Abhyankar [1], proven by Jung [10] for germs of complex analytic surfaces in 1887) Let $R \to S$ be finite where R is a regular algebraic local ring and S is normal. Suppose that the discriminant of R in S is a monomial in some regular system of parameters $x_1, \ldots, x_d$ of R. Further suppose that the extensions of Dedekind domains $R_{(x_i)} \to S_{(x_i)}$ are tamely ramified. Then the completion at any maximal ideal of S has the form $\hat{S} = k[[M_1, \ldots, M_r]]$ where M_i are monomials in roots of $x_1, \ldots, x_d$.*

In characteristic zero, all finite extensions are tamely ramified. However, Abhyankar gave examples of wildly ramified extensions (in positive characteristic) where the conclusions of the theorem do not hold.

We now give a quick explanation of Jung's method to show that in characteristic zero, local embedded resolution in dimension d implies local uniformization in dimension d. Let X be a d-dimensional algebraic variety of characteristic zero. Locally project X by a finite map onto a nonsingular variety Y of the same dimension d. The discriminant Δ of the map has dimension $d-1$. By embedded resolution of singularities in dimension d, we have a birational map $\varphi : Z \to Y$ such that Z is nonsingular and $\varphi^*(\Delta)$ has simple normal crossings. Let W be the normalization of Z in the function field K of X. The discriminant of $W \to Z$ is $\varphi^*(\Delta)$ so by the Abhyankar-Jung Theorem (we are in characteristic zero) W has very simple singularities which can be resolved by a simple combinatorial algorithm.

This method breaks down in positive characteristic, when there is wild ramification in the local projection $X \to Y$.

Abhyankar's PhD thesis [2] proved resolution of surface singularities in positive characteristic. He proved local uniformization in two dimensional algebraic function fields, from which resolution of surfaces follows as discussed earlier. Abhyankar used the ramification theory of valuations to reduce to the case of an Artin-Schreier polynomial, that is a polynomial $z^p + g^{p-1}z + f = 0$ where g and f are polynomials in x and y. He then gives a difficult algorithm to prove resolution of singularities along a valuation dominating the algebraic local ring $k[x, y, z]/(z^p + g^{p-1}z + f)$

Using his later algorithm [3] proving embedded resolution in dimension three, Abhyankar modified Jung's method to prove resolution of singularities of 3-folds in characteristic $p > 5$. The proof begins by adapting a method of Albanese [4] from 1924 to show that if X is a variety of dimension d, then there is birational projective map from X to a variety Y such that all local rings of Y have multiplicity $\leq d!$. Abhyankar applies this when $d = 3$. At every point α of Y , there is a local projection to a nonsingular variety Z, of degree less than or equal to the multiplicity of α which is less than or equal to $6 = 3! < p$ where the last inequality is our assumption on the characteristic p of k. With this degree restriction, the Abhyankar-Jung Theorem works, as there can be no wild ramification in extensions of degree less than p. This gives the proof of local uniformization, so that resolution of singularities is true since $d = 3$.

The above argument would prove local uniformization in dimension four and characteristic $p > 4! = 24$ if embedded resolution of singularities in dimension 4 were known to be true.

The recent proof of Cossart and Piltant [6] which proves resolution of singularities in dimension 3 and in all characteristics, begins with the observation that it is only necessary to prove local uniformization since the dimension is three. They use methods of Abhyankar to reduce to an Artin-Schreier extension. Embedded resolution in dimension three plays an essential role in this reduction. Then an extraordinarily long and technical argument is used to resolve the Artin-Schreier extension along a valuation. It is not known how to extend this argument, even to an extension of degree p^2. This is the first open case in proving embedded resolution in dimension four.

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TMC Distinguished Lecture Series completes one year

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The last two issues of the Bulletin contained brief write-ups on the TMC Distinguished Lecture Series (in short, TMC DLS), and the lectures in this series by Professors Yves Benoist (Paris), Bernd Sturmfels(Leipzig/Berkeley), Nalini Anantharaman (Strasbourg), and Alex Lubotzky (Jerusalem) as well as by Professors Scott Sheffield (MIT), Karen Smith (Michigan), Daniel Wise (Montreal), Mikhail Lyubich (Stoney Brook), and Tadeshi Tokieda (Stanford). This activity, which began in October 2020, has now completed one year. It continues to grow strong and is able to feature virtual talks almost every month by some of the best researchers and expositors around the world, thanks to the efforts of the Scientific Committee and the Organizing Committee.

In the recent past, the TMC DLS has featured talks by **Prof. Siddharth Mishra** (ETH, Zürich, Switzerland) on Deep learning and computation of PDEs, **Prof. Noga Alon** (Tel Aviv University, Israel and Princeton University, USA) on The Necklace Theorem: challenges, variants and algorithms, and **Prof. Mladen Bestvina** (University of Utah, USA) on Asymptotic dimension.

The TMC DLS is organized by The (Indian) Mathematics Consortium, and it is co-hosted by IIT Bombay and ICTS-TIFR Bengaluru. More information about the TMC DLS is available at: *https://sites.google.com/view/distinguishedlectureseries/*

The videos of the talks held thus far are available on the TMC YouTube Channel at *https://www.youtube.com/channel/UCoarOpo_-9fgzFWDap6dFFw/*

2. Chinese Remainder Theorem and Interpolation Formulae

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Abstract: In this article we shall show how Lagrange Interpolation Formula can be considered as an analogue of Chinese Remainder Theorem. Using the same idea we shall obtain a formula for a curve passing through given finitely many points and also having given tangent directions at those points. This gives us a version of Hermite Interpolation Formula. We then indicate how the interpolation conditions in terms of higher order derivatives too can be tackled using the Polynomial Version of Chinese Remainder Theorem. These results have applications in Computer Aided Geometric Design (CAGD).

2.1 Chinese Remainder Theorem for Integers

The Chinese Remainder Theorem (CRT) tells us that given positive integers $m_1, \ldots, m_r$, coprime in pairs, and given integers $x_i, 0 \le x_i < m_i$, there is a nonnegative integer which leaves remainders x_i when divided by $m_i, 1 \le i \le r$. More precisely, in Gauss's congruence notation we express this result as

Theorem 1: (Chinese Remainder Theorem). Given positive integers $m_1, \ldots, m_r$, coprime in pairs, and given integers x_i, the system of congruences

$$x \equiv x_i \pmod{m_i}, \; i = 1, 2, \ldots, r, \tag{2.1}$$

has a unique solution modulo $m = m_1 m_2 \cdots m_r$. The unique solution to the above system of congruences is given by

$$x \equiv x_1 M_1 \overline{M}_1 + x_2 M_2 \overline{M}_2 + \cdots + x_r M_r \overline{M}_r \pmod{m} \tag{2.2}$$

where $m = m_1 m_2 \ldots m_r, M_i = \dfrac{m}{m_i}$, and $\overline{M}_i$ is the inverse of $M_i \pmod{m_i}, 1 \le i \le r$.

The theorem is in the context of the ring of integers. The theorem says that an integer solution to the system of congruences exists and any two integer solutions differ by an integer multiple of m. More details and a proof of CRT can be found in any book on Elementary Number Theory e.g. [3], [4].

Note that an integer $\overline{M}_i$ exists as an inverse of $M_i \pmod{m_i}$ and is unique $\pmod{m_i}$. For this observe that $m_1, m_2, \ldots, m_r$ are coprime in pairs, so M_i is relatively prime to m_i.

Observe that in (2), $M_1 \overline{M}_1 \equiv 1 \pmod{m_1}$ whereas $M_1 \equiv 0 \pmod{m_j}$ for $j \ne 1$.
Hence, $x \equiv x_1 \pmod{m_1}$. Similarly $x \equiv x_i \pmod{m_i}$ for all i.
This shows that (2) gives a solution to the system (1).

Also, if x and x' are integer solutions of (1), then $x \equiv x_1 \pmod{m_1}$ and $x' \equiv x_1 \pmod{m_1}$.
Hence $x \equiv x' \pmod{m_1}$. Similarly, $x \equiv x' \pmod{m_i}$ for all i.
As $m_1, m_2, \ldots, m_r$ are coprime in pairs, we get $x \equiv x' \pmod{m_1 m_2 \ldots m_r}$. Thus the solution is unique modulo $m_1 m_2 \ldots m_r$.

In (2), we require to find the inverse $\overline{M}_i$ of $M_i \pmod{m_i}$.
This can be done by trial for small numbers, and for large numbers we can use Euclid's algorithm. Indian Mathematicians such as Aryabhata (b. 476) had developed an algorithm called Kuttaka, using which also the inverse can be obtained.

Talk at Workshop on "Algebra, Number Theory and their Applications" organised jointly with The Mathematics Consortium during 13th-14th March, 2020, by Dept of Mathematics, Ramakrishna Mission Vivekananda Centenary College, Rahara, Kolkata – 700118.

2.2 An example of CRT

Solve the system of congruences

$$x \equiv 2 \pmod 3,$$
$$x \equiv 3 \pmod 5,$$
$$x \equiv 5 \pmod 7.$$

Here we intend to find all integers which leave remainders 2, 3, 5 when divided by 3, 5, 7 respectively.

Solution: As per the notation in (1), $m_1 = 3, m_2 = 5, m_3 = 7; m = 3 \cdot 5 \cdot 7 = 105.$

$$M_1 = 5 \cdot 7 = 35, M_2 = 3 \cdot 7 = 21, M_3 = 3 \cdot 5 = 15.\ x_1 = 2, x_2 = 3, x_3 = 5.$$

To find the inverse $\overline{M}_1$ of $M_1 \pmod{m_1}$ it will be good to reduce $M_1 \pmod{m_1}$. Hence $M_1 = 35 \equiv 2 \pmod 3$. Similarly, $M_2 = 21 \equiv 1 \pmod 5$ and $M_3 = 15 \equiv 1 \pmod 7$.
We can thus take $\overline{M}_1 = 2, \overline{M}_2 = 1, \overline{M}_3 = 1$. (Here the inverses coincide with the reduced numbers, but that is not usually the case.) Now by (2),

$$x \equiv x_1 M_1 \overline{M}_1 + x_2 M_2 \overline{M}_2 + x_3 M_3 \overline{M}_3 \equiv 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 5 \cdot 15 \cdot 1 \equiv 140 + 63 + 75 \equiv 68 \pmod{105}.$$

Thus $x \equiv 68 \pmod{105}$ is the common solution of the system and all integer solutions are $x = 68 + 105k, k \in \mathbb{Z}$.

Similar questions can be considered for polynomials too. For example, we may wish to find polynomials with real coefficients having given remainders when they are divided by given polynomials.

2.3 CRT for Polynomial

We can easily state the version of CRT for polynomials.
Theorem 2 (Chinese Remainder Theorem for Polynomials): Let $m_1(x), \ldots, m_r(x)$ be non-constant polynomials with real coefficients, such that $m_1(x), \ldots, m_r(x)$ are relatively prime in pairs, i.e. any two have gcd 1. Let $f_1(x), \ldots, f_r(x)$ be given polynomials with real coefficients. Then the system of polynomial congruences

$$f(x) \equiv f_i(x) \pmod{m_i(x)}, \text{for } 1 \le i \le r,$$

has a unique solution modulo the product $m_1(x)m_2(x)\ldots m_r(x)$.
This can be equivalently stated in the language of remainders as:

Given nonconstant polynomials $m_1(x), \ldots, m_r(x)$ of degrees $d_1, \ldots, d_r$ and relatively prime in pairs, and polynomials $f_i(x)$ of degree $< \deg(m_i(x))$, $1 \le i \le r$, there is a unique polynomial $f(x)$ of degree $< \sum_{i=1}^{r} d_i$ which leaves remainder $f_i(x)$ when divided by $m_i(x), 1 \le i \le r$.

Here again the solution is

$$f(x) \equiv \sum_{i=1}^{r} f_i(x) M_i(x) \overline{M}_i(x) \pmod{m(x)} \tag{2.3}$$

where $M_i(x) = \dfrac{m(x)}{m_i(x)}$, $m(x)$ being the product $m_1(x) \ldots m_r(x)$.
Also $\overline{M}_i(x)$ is a polynomial which works as an inverse of $M_i(x)$ modulo $m_i(x)$.

The proof of Polynomial Version of CRT is analogous to the proof of Integer CRT. It can be easily verified that the solution is unique modulo $m(x)$. From one solution we get all solutions by adding polynomial multiples of $m(x)$.

Note that a polynomial $\overline{M}_i(x)$ can be found by trial or it can be found by using Euclid's algorithm. Since $M_i(x)$ is relatively prime to $m_i(x)$, by Euclid's algorithm for polynomials, the gcd 1 is a polynomial linear combination of $M_i(x)$ and $m_i(x)$.
Thus there exist polynomials $\overline{M}_i(x)$ and $h_i(x)$ with real coefficients such that

$$1 = M_i(x)\overline{M}_i(x) + m_i(x)h_i(x), \quad 1 \le i \le r.$$

This gives $\overline{M}_i(x)$ as the inverse of $M_i(x) \pmod{m_i(x)}$.

We can as well take $\overline{M}_i(x)$ to be a polynomial of degree $< \deg(m_i(x))$. Then, $\overline{M}_i(x)$ is uniquely determined. If $\deg(\overline{M}_i(x)) \ge \deg(m_i(x))$, we replace it by the remainder when $\overline{M}_i(x)$ is divided by $m_i(x)$. Observe that if $m_i(x)$ is a linear polynomial, then $\overline{M}_i(x)$ can be taken to be just a constant polynomial.

2.4 Lagrange's Interpolation Formula as a particular case of Polynomial CRT

We now take $m_i(x)$ as linear polynomials $x - x_i, 0 \le i \le n$, x_i distinct. Thus $x - x_i$ are coprime in pairs.

We require a polynomial $f(x)$ which has value y_i at $x = x_i$, $0 \le i \le n$.
By Remainder Theorem, y_i is the remainder when $f(x)$ is divided by $x - x_i$, or, equivalently $f(x) \equiv y_i \pmod{x - x_i}$. Thus we need to solve the system of $n+1$ polynomial congruences

$$f(x) \equiv y_i \pmod{x - x_i},\ 0 \le i \le n.$$

By PCRT (Polynomial CRT) the common solution is

$$f(x) \equiv \sum_{i=0}^{n} y_i M_i(x)\overline{M}_i(x) \pmod{m(x)},$$

where $\mathbf{m}(x) = (x - x_0)\dots(x - x_n)$ and $M_i(x) = \dfrac{m(x)}{x - x_i} = \prod_{j \ne i}(x - x_j)$.

We sometimes write $M_i(x) = (x - x_0)\dots\overbrace{(x - x_i)}\dots(x - x_n)$ where $\frown$ denotes that the term below it viz. $x - x_i$ is omitted.
We have now to find $\overline{M}_i(x)$. This can be taken to be a constant polynomial, i.e. a constant say c_i, since the remainder when any polynomial is divided by $x - x_i$ is a constant.
We require $c_i M_i(x) \equiv 1 \pmod{x - x_i}$. Thus 1 is the remainder when $c_i M_i(x)$ is divided by $x - x_i$.
By Remainder Theorem, putting $x = x_i$ we get $c_i M_i(x_i) = 1$ or $c_i = \dfrac{1}{M_i(x_i)}$.
Hence

$$\begin{aligned} f(x) &\equiv \sum_{i=0}^{n} y_i M_i(x) \cdot c_i \pmod{m(x)} \\ &\equiv \sum_{i=0}^{n} y_i \frac{M_i(x)}{M_i(x_i)} \pmod{m(x)}. \end{aligned}$$

We thus get
Theorem 3 (Lagrange's Interpolation Formula). The smallest degree polynomial satisfying $f(x_i) = y_i, \quad 0 \le i \le n$, is

$$f(x) = \sum_{i=0}^{n} y_i \frac{(x - x_0)\dots\overbrace{(x - x_i)}\dots(x - x_n)}{(x_i - x_0)\dots\overbrace{(x_i - x_i)}\dots(x_i - x_n)}.$$

All the polynomials satisfying $f(x_i) = y_i, 0 \leq i \leq n$ are given by $f(x) + m(x)g(x)$ where $f(x)$, $m(x)$ are as given above and $g(x)$ is any polynomial over $\mathbb{R}$.

We thus see that Lagrange's Interpolation Formula is the polynomial version of CRT when the moduli are restricted to linear polynomials. The curve $y = f(x)$ is thus a curve of degree $\leq n$ passing through (x_i, y_i), $0 \leq i \leq n$.

Observe that the $n+1$ Lagrange Polynomials $L_i(x) = \dfrac{M_i(x)}{M_i(x_i)}, 0 \leq i \leq n$, form a basis of the $\mathbb{R}$-vector space of polynomials of degree $\leq n$, and when we express any polynomial $f(x)$ of degree $\leq n$ as a linear combination of the basis vectors $L_i(x)$, the coefficients are nothing but the values of $f(x)$ at $x = x_i$. Thus any polynomial $f(x)$ of degree $\leq n$ can be written as $f(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$. Dividing by $\prod_{i=0}^{n}(x - x_i)$, we get

Corollary (Partial Fractions). If $f(x)$ is a polynomial of degree $\leq n$, then for distinct $x_0, x_1, \cdots, x_n$,

$$\frac{f(x)}{\prod_{i=0}^{n}(x - x_i)} = \sum_{i=0}^{n} \frac{f(x_i)/M_i(x_i)}{x - x_i}.$$

2.5 Hermite Interpolation

Hermite Interpolating polynomial is a polynomial curve $y = f(x)$ passing through the points (x_i, y_i) and having given tangent directions y_i' at (x_i, y_i).

Observe that any polynomial $f(x)$ satisfying $f(x_i) = y_i$ is of the form $f(x) = y_i + (x - x_i)g_i(x),\ g_i(x) \in \mathbb{R}[x]$. If we further want that $f'(x_i) = y_i'$, then $f(x)$ takes the form

$$f(x) = y_i + y_i'(x - x_i) + (x - x_i)^2 h_i(x), \quad h_i(x) \in \mathbb{R}[x].$$

In other words $f(x)$ satisfies the congruence

$$f(x) \equiv y_i + y_i'(x - x_i) \pmod{(x - x_i)^2}.$$

Thus if we wish to have a polynomial $f(x)$ such that $f(x_i) = y_i$ and $f'(x_i) = y_i',\ 0 \leq i \leq n$, we proceed to find the solution to the system of polynomial congruences

$$f(x) \equiv y_i + y_i'(x - x_i) \pmod{(x - x_i)^2}, \quad 0 \leq i \leq n.$$

By PCRT, using (3) we let $\mathbf{m}_i(x) = (x - x_i)^2,\ m(x) = \prod_{i=0}^{n}(x - x_i)^2,\ \mathcal{M}_i(x) = \dfrac{\mathbf{m}(x)}{(x - x_i)^2}$.

We want to find $\overline{\mathcal{M}}_i(x)$ such that $\mathcal{M}_i(x)\overline{\mathcal{M}}_i(x) \equiv 1 \mod (x - x_i)^2$. $\overline{\mathcal{M}}_i(x)$ can be taken to be a polynomial of degree ≤ 1, which we take in the form $\overline{\mathcal{M}}_i(x) = a_i + b_i(x - x_i)$.

Thus $\mathcal{M}_i(x)[a_i + b_i(x - x_i)] \equiv 1 \mod (x - x_i)^2$ or $\mathcal{M}_i(x)[a_i + b_i(x - x_i)] = 1 + g_i(x)(x - x_i)^2$, for some $g_i(x) \in \mathbb{R}[x]$.

Putting $x = x_i$, we get $a_i = 1/\mathcal{M}_i(x_i)$. Differentiating and then putting $x = x_i$, we get

$$\left(\frac{d}{dx}(\mathcal{M}_i(x)\right)_{x=x_i} [a_i + b_i(x - x_i)]_{x=x_i} + b_i\mathcal{M}_i(x_i) = 0.$$

So

$$a_i\left(\frac{d}{dx}\mathcal{M}_i(x)\right)_{x=x_i} + b_i\mathcal{M}_i(x_i) = 0. \tag{2.4}$$

Now $\mathcal{M}_i(x) = \prod_{j \neq i}(x - x_j)^2$.

So $\dfrac{d}{dx}\mathcal{M}_i(x) = \sum_{j \neq i} \left\{ 2(x - x_j) \prod_{k \neq j,i} (x - x_k)^2 \right\}$, so that

$$\begin{aligned}
\left(\frac{d}{dx}\mathcal{M}_i(x)\right)_{x=x_i} &= \sum_{j \neq i} \left\{ 2(x_i - x_j) \prod_{k \neq j,\, i} (x_i - x_k)^2 \right\} \\
&= 2\sum_{j \neq i} \left\{ \frac{(x_i - x_j)^2}{(x_i - x_j)} \prod_{k \neq j,\, i} (x_i - x_k)^2 \right\} \\
&= 2\sum_{j \neq i} \frac{\mathcal{M}_i(x_i)}{x_i - x_j}.
\end{aligned}$$

Now by (4),

$$\frac{1}{\mathcal{M}_i(x_i)} \cdot 2\sum_{j \neq i} \frac{\mathcal{M}_i(x_i)}{x_i - x_j} + b_i \mathcal{M}_i(x_i) = 0.$$

Hence cancelling $\mathcal{M}_i(x_i)$ we get $b_i = -\dfrac{2}{\mathcal{M}_i(x_i)} \sum_{j \neq i} \dfrac{1}{x_i - x_j}$.

Thus

$$\begin{aligned}
\overline{\mathcal{M}}_i(x) &= a_i + b_i(x - x_i) \\
&= \frac{1}{\mathcal{M}_i(x_i)} - \frac{2}{\mathcal{M}_i(x_i)} \sum_{j \neq i} \frac{1}{x_i - x_j}(x - x_i).
\end{aligned}$$

Using (3), taking $f_i(x) = y_i + y_i'(x - x_i)$ we get

$$f(x) \equiv \sum_{i=0}^{n} \left[y_i + y_i'(x - x_i) \right] \mathcal{M}_i(x) . \frac{1}{\mathcal{M}_i(x_i)} \left[1 - 2(x - x_i) \sum_{j \neq i} \frac{1}{x_i - x_j} \right] \pmod{\mathbf{m}(x)}.$$

Since $(x - x_i)^2 \mathcal{M}_i(x) = \mathbf{m}(x)$, we get

$$f(x) \equiv \sum_{i=0}^{n} \frac{\mathcal{M}_i(x)}{\mathcal{M}_i(x_i)} \left[y_i + y_i'(x - x_i) - 2y_i(x - x_i) \sum_{j \neq i} \frac{1}{x_i - x_j} \right] \pmod{\mathbf{m}(x)}.$$

Let, for $L_i(x)$ as in Section 4,

$$\mathcal{L}_i(x) = \frac{\mathcal{M}_i(x)}{\mathcal{M}_i(x_i)} = \frac{\prod_{j \neq i}(x - x_j)^2}{\prod_{j \neq i}(x_i - x_j)^2} = L_i^2(x), \text{ and } \beta_i = y_i' - 2y_i \sum_{j \neq i} \frac{1}{x_i - x_j}. \tag{2.5}$$

Then $f(x) \equiv \sum_{i=0}^{n} [y_i + \beta_i(x - x_i)] \mathcal{L}_i(x) \pmod{\mathbf{m}(x)}$. By degree considerations, the smallest degree polynomial satisfying this congruence is the polynomial on RHS. We thus get

Theorem 4: The polynomial $f(x)$ of smallest degree satisfying $f(x_i) = y_i$ and $f'(x_i) = y_i'$ for $0 \leq i \leq n$ is the polynomial

$$f(x) = \sum_{i=0}^{n} [y_i + \beta_i(x - x_i)] \mathcal{L}_i(x), \text{ where } \mathcal{L}_i(x) \text{ and } \beta_i \text{ are given by (5).}$$

All polynomials satisfying these conditions are $f(x) + g(x)\mathbf{m}(x)$ for $g(x) \in \mathbb{R}[x]$.

This also gives the curve $y = f(x)$ passing through (x_i, y_i) and having tangent direction y_i' at (x_i, y_i), $0 \leq i \leq n$.

The polynomial $f(x)$ in the statement of the theorem lies in the vector space of polynomials of degree $\leq 2n+1$ expressed in terms of the basis $\{\mathcal{L}_i(x), (x-x_i)\mathcal{L}_i(x)\}, 0 \leq i \leq n$, and is uniquely determined. As in the Lagrange's Interpolation Formula, here also, all polynomials satisfying the given conditions about the value and derivative of the function at $n+1$ points are given by $f(x) + g(x)\mathbf{m}(x)$ where $g(x) \in \mathbb{R}[x]$.

Application to partial fractions.

If $f(x)$ is a polynomial in $\mathbb{R}[x]$ of degree $\leq 2n+1$, then for distinct $x_i, 0 \leq i \leq n$, we get the partial fraction expansion

$$\frac{f(x)}{\prod_{i=0}^n (x-x_i)^2} = \sum_{i=0}^n \left[\frac{y_i / \mathcal{M}_i(x_i)}{(x-x_i)^2} + \frac{\beta_i / \mathcal{M}_i(x_i)}{x-x_i} \right],$$

where $f(x_i) = y_i, f'(x_i) = y_i'$ and $\beta_i = y_i' - 2y_i \sum_{j \neq i} \frac{1}{x_i - x_j}$.

Historically, Charles Hermite (1822-1901) obtained the Interpolation Formula for a function having given values and derivatives at $n+1$ points. For a version of Hermite's Interpolation Formula given below and its proof, see [2], Theorem 3.9.

Theorem 5 (Hermite's Interpolation Formula). The unique polynomial of least degree having values y_i at x_i and having derivative y_i' at x_i, $0 \leq i \leq n$, is the Hermite Interpolation Polynomial of degree at most $2n+1$ given by

$$H_{2n+1} = \sum_{i=0}^n y_i H_{n,i}(x) + \sum_{i=0}^n y_i' \hat{H}_{n,i}(x),$$

where for $L_{n,i} = L_i$ as in Section 4,

$$H_{n,i}(x) = [1 - 2(x-x_i)L_{n,i}'(x_i)]L_{n,i}^2(x) \text{ and } \hat{H}_{n,i}(x) = (x-x_i)L_{n,i}^2(x).$$

Proof: One checks that $H_{n,i}(x)$ is a polynomial which is 1 at $x = x_i$ and 0 at all $x_j, j \neq i$. Further, its derivative vanishes at all x_j. On the other hand, the polynomial $\hat{H}_{n,i}(x)$ has its derivative 1 at $x = x_i$ and 0 at $x = x_j, j \neq i$. Further, it vanishes at $x = x_j$, for all j. H_{2n+1}, being of degree $\leq 2n+1$, is uniquely determined by the given conditions.

Alternatively, by the calculation after (4), $\mathcal{L}_i'(x) = \frac{\mathcal{M}_i'(x)}{\mathcal{M}_i(x_i)}$, so $\mathcal{L}_i'(x_i) = 2\sum_{j \neq i} \frac{1}{x_i - x_j}$. Since $\mathcal{L}_i(x) = L_i^2(x)$, we get $\mathcal{L}_i'(x_i) = 2L_i(x_i)L_i'(x_i)$, so (or by direct calculation), $L_i'(x_i) = \sum_{j \neq i} \frac{1}{x_i - x_j}$. Hence Theorem 4 gives Theorem 5.

Another formulation given in [1] is:

Theorem 6 (Hermite Interpolation Formula) For $m(x)$ and $L_i(x)$ as in Section 4,

$$H_{2n+1}(x) = \sum_{i=0}^n \left\{ y_i \left[1 - \frac{m''(x_i)}{m'(x_i)}(x-x_i) \right] + y_i'(x-x_i) \right\} L_i^2(x).$$

Proof. This follows from our Theorem 4, since $\frac{m''(x_i)}{m'(x_i)}$ can be shown to be equal to $2\sum_{j \neq i} \frac{1}{x_i - x_j}$ in the expression for β_i in Theorem 3. For this we proceed as follows:

$m(x) = (x-x_0)\ldots(x-x_n)$ and $M_i(x) = \prod_{j \neq i}(x-x_j)$, so, $m'(x) = \sum_{j=0}^n M_j(x)$.

Also $m''(x) = \sum_{j=0}^n M_j'(x)$. $M_j'(x) = \sum_{k \neq j} \prod_{l \neq k,j}(x-x_l)$.

$m'(x_i) = M_i(x_i)$. $M_j'(x_i) = \sum_{k \neq j} \prod_{l \neq k,j}(x_i - x_l)$.

Suppose $j = i$. Then $M_i'(x_i) = \sum_{k \neq i} \prod_{l \neq k,i}(x_i - x_l)$

$$= \sum_{k \neq i} \frac{\prod_{l \neq i}(x_i - x_l)}{x_i - x_k} = \sum_{k \neq i} \frac{M_i(x_i)}{x_i - x_k} = M_i(x_i) \sum_{k \neq i} \frac{1}{x_i - x_k}.$$

Now, for $j \neq i$, $M_j'(x_i) = \sum_{k \neq j} \prod_{l \neq k,j}(x_i - x_l)$.
Here in the sum, every term for $k \neq i$ can have $l = i$ in the product, so as to get a contribution $x_i - x_i = 0$. Thus in the sum only the term with $k = i$ survives. Hence $M_j'(x_i) = \prod_{l \neq i,j}(x_i - x_l) = \dfrac{M_i(x_i)}{x_i - x_j}$.

Hence, $\sum_{j \neq i} M_j'(x_i) = \sum_{j \neq i} \dfrac{M_i(x_i)}{x_i - x_j} = M_i(x_i) \sum_{j \neq i} \dfrac{1}{x_i - x_j}$.

Adding the results for $j = i$ and $j \neq i$ we get $m''(x_i) = \sum_{j=0}^{n} M_j'(x_i) = 2M_i(x_i) \sum_{j \neq i} \dfrac{1}{x_i - x_j}$.

Since $m'(x_i) = M_i(x_i)$, we get, $\dfrac{m''(x_i)}{m'(x_i)} = 2 \sum_{j \neq i} \dfrac{1}{x_i - x_j}$.

We thus see that Theorem 4 gives a version of Hermite's Interpolation Formula and it is in a simplified form as compared to Theorems 5 and 6.
Note: The Hermite Interpolation Polynomials are not to be confused with Hermite polynomials which form a classical sequence of orthogonal polynomials.

2.6 Higher Interpolation

Using Polynomial CRT, it is possible to find a polynomial $f(x)$ such that f, f', f'' (or higher derivatives) have given values at x_i, $0 \leq i \leq n$. Also information for different points can be upto derivatives of different orders. For this, note that by Taylor's formula, a polynomial $f(x)$ of degree $\leq n$ can be expanded in powers of $x - a$ as

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n.$$

In this context we have the following
Theorem 7. A polynomial $f(x)$ having values of $f(x_i), f'(x_i), \ldots, f^{(\mu_i)}(x_i), 0 \leq i \leq n$, as $y_i, y_i', \ldots, y_i^{(\mu_i)}$, respectively, is a solution to the system of polynomial congruences

$$f(x) \equiv A_i(x) \pmod{(x - x_i)^{\mu_i + 1}}, 0 \leq i \leq n, \tag{2.6}$$

where $A_i(x) = y_i + y_i'(x - x_i) + \frac{1}{2!}y_i''(x - x_i)^2 + \ldots + \frac{1}{\mu_i!}y^{(\mu_i)}(x - x_i)^{\mu_i}$. It is given by $f(x) \equiv \sum A_i(x)M_i(x)\overline{M}_i(x) \pmod{m(x)}$, where $m(x) = \prod_{i=0}^{n}(x - x_i)^{\mu_i+1}, M_i(x) = \dfrac{m(x)}{(x - x_i)^{\mu_i+1}}$, and $\overline{M}_i(x)$ is the inverse of $M_i(x) \pmod{(x - x_i)^{\mu_i+1}}$. The polynomial of smallest degree satisfying the given conditions is of degree $\leq \sum_{i=0}^{n} \mu_i + n$ and is uniquely determined.

Proof. The proof follows by converting the given conditions in the form of the congruence condition (6) and then using Polynomial CRT.

Interested readers may refer to [1] for results about Higher Interpolation formulae obtained in a different way. Such results have applications in Computer Aided Geometric Design (CAGD).
An Example.

We illustrate the method of Chinese Remainder Theorem used for Higher Interpolation in this example.

We wish to find a polynomial $f(x)$ of least degree satisfying:

(i) $f(-1) = 1$,

(ii) $f(0) = -1, f'(0) = -3,\ f''(0) = 0$,

(iii) $f(1) = -3, f'(1) = 2$.

Note that here there are 6 conditions and the polynomial of least degree satisfying these conditions is of degree ≤ 5 and is uniquely determined. We take 3 polynomials $m_i(x), i = 1,2,3$ as $m_1(x) = x+1, m_2(x) = x^3, m_3(x) = (x-1)^2$, so that the given conditions can be written in the congruence notation as

$$f(x) \equiv A_i(x) \pmod{m_i(x)}, i = 1,2,3,$$

where $A_1(x) = 1, A_2(x) = -1-3x$, and $A_3(x) = -3+2(x-1) = -5+2x$.

We have $M_1(x) = x^3(x-1)^2, M_2(x) = (x+1)(x-1)^2, M_3(x) = (x+1)x^3$.

We need to find the inverses $\overline{M}_i(x)$ of $M_i(x)$ modulo $m_i(x)$, $i = 1,2,3$. For this, to simplify the matter, we first reduce each $M_i(x)$ modulo $m_i(x)$. Note that to find the inverse of a polynomial modulo $(x-a)^k$, it is convenient to write the given polynomial in powers of $x-a$ and use the idea of a geometric series.

1 Since $m_1(x) = x+1$ is linear, we have, by Remainder theorem, $M_1(x) \equiv M_1(-1) \pmod{x+1}$. Thus $M_1(x) \equiv -4 \pmod{x+1}$. Hence $\overline{M}_1(x) \equiv -\dfrac{1}{4} \pmod{x+1}$

2 $M_2(x) = (x+1)(x-1)^2 = (x^2-1)(x-1) = x^3 - x^2 - x + 1 \equiv -x^2 - x + 1 \pmod{x^3}$. The inverse $\pmod{x^3}$ can be obtained by using $(1-y)(1+y+y^2) = 1-y^3$ and putting $y = x+x^2$. Here RHS $\equiv 1 \pmod{x^3}$ and

$$1+y+y^2 = 1+(x+x^2)+(x+x^2)^2 \equiv 1+x+2x^2 \pmod{x^3}.$$

Thus $\overline{M}_2(x) \equiv 1+x+2x^2 \pmod{x^3}$.

3 $M_3(x) = (x+1)x^3$. To reduce mod $(x-1)^2$ and then find inverse it will be convenient to write $M_3(x)$ in powers of $x-1$. Thus we have

$$x^3 = [(x-1)+1]^3 \equiv 3(x-1)+1 \pmod{(x-1)^2}$$

Hence $M_3(x) = (x+1)x^3 \equiv [(x-1)+2][3(x-1)+1] \equiv 7(x-1)+2 \pmod{(x-1)^2}$.
Now, $[7(x-1)+2][7(x-1)-2] = 49(x-1)^2 - 4 \equiv -4 \pmod{(x-1)^2}$.
Hence $\overline{M}_3(x) \equiv -\frac{1}{4}[7(x-1)-2] \equiv -\frac{1}{4}(7x-9) \pmod{(x-1)^2}$. Now,

$$m(x) = m_1(x)m_2(x)m_3(x) = (x+1)x^3(x-1)^2 = x^6 - x^5 - x^4 + x^3.$$

So $x^6 \equiv x^5 + x^4 - x^3 \pmod{m(x)}$. Using this and simplifying we get

$$\sum_{i=1}^{3} A_i(x)M_i(x)\overline{M}_i(x) \equiv x^5 - 3x - 1 \pmod{m(x)}.$$

This gives us $x^5 - 3x - 1$ as the unique polynomial of least degree satisfying the given conditions.
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3. A note on Koebe's General Uniformisation Theorem

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Dedicated to the memory of Professor R. R. Simha

Abstract: The uniformisation theorem which is also known as the Riemann mapping theorem for Riemann surfaces played a very significant role in the history of modern Mathematics. Several leading mathematicians of the 19th century contributed enormous amount of Mathematics, with fresh innovations, to supply a complete proof. It created bridges between Complex function theory to other topics, and opened a common space in mathematics to unite geometric function theory and topology. We see that Koebe's General Uniformisation Theorem is a more general theorem than the Uniformisation Theorem (Riemann mapping theorem for Riemann surfaces) and the latter can be easily deduced from Koebe's General Uniformisation Theorem.
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3.1 Introduction

According to Ahlfors, "Uniformisation Theorem" for Riemann surfaces was "perhaps the single most important theorem in the whole theory of analytic functions". It originated in the works of Riemann, Poincaré, and Klein, and influenced several mathematicians during the nineteenth century. It created bridges between Complex function theory to other topics and opened a common space in mathematics to unite geometric function theory and topology.

In modern language the theorem may be stated as follows:

Theorem 1. *Every simply connected Riemann surface is biholomorphic to one of the following:* The Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$.

The **Uniformisation Theorem** characterizes, up to analytical equivalence, all the simply connected Riemann surfaces, or in other words, it characterizes universal covering spaces of Riemann surfaces. So this is also called the Riemann mapping theorem for Riemann surfaces.

In 1900, at the International Congress of Mathematicians (ICM) Hilbert stated the uniformisation theorem as 22^{nd} problem in the list of his 23 problems. In 1907, Koebe and Poincaré independently published rigorous proofs of the Uniformisation theorem. The deepest form of the uniformisation was proved by Koebe, and is called **Koebe's General Uniformisation Theorem for planar Riemann surfaces**, or Koebe's planarity theorem.

On any Riemann surface X the following two conditions are equivalent:

1. (Analytical) Every smooth closed 1-form with compact support on X is exact;
2. (Topological) $X \setminus \{\gamma\}$ is disconnected for any Jordan curve $\{\gamma\}$ on X. (In fact it is disconnected in to two components).

A Riemann surface X is called **planar** or ***schlichtartig*** if either of the above conditions is satisfied.

Here are some examples of Riemann surfaces that are planar: Any simply connected Riemann surface is planar: On such a surface, every smooth closed 1-form is exact, by Green's theorem. In particular $\hat{\mathbb{C}}$ and $\mathbb{C}$ are planar.

Suppose X is a planar Riemann surface and Y is an open subset of X. Then it is easy to see that Y is planar. Thus, in particular, all domains Ω in $\hat{\mathbb{C}}$ (or $\mathbb{C}$) are planar. This justifies the nomenclature.

Remark 2. *The above discussion applies also to general smooth two-dimensional manifolds and the property of planarity is invariant under diffeomorphisms.*

Koebe's theorem characterises Riemann surfaces that are biholomorphic to domains in the Riemann sphere $\hat{\mathbb{C}}$. It states that

Theorem 3. *(Koebe 1909) (Koebe's General Uniformisation Theorem) Every planar Riemann surface X is biholomorphic to a domain in the Riemann sphere.*

Koebe wrote his doctoral dissertation in 1905 under the supervision of Schwarz. He published several papers in this area that brought him the attention of leading mathematicians Poincaré, Hilbert and others. As a result he was invited to speak at the International Congress of Mathematicians, held in Rome in 1908. In fact Koebe spent the rest of his scientific life studying the Uniformisation Theorem from all directions.

We point out that while many proofs of the uniformisation theorem have appeared over the last century (and are still appearing), Koebe's more general theorem was not even stated in the majority of text books on Riemann surfaces.

In Nevanlinna's classic work [7], a proof attributed to Koebe and Hilbert is given, but appears to be rather formidable. We should also point out that Nevanlinna's proof assumes that the potential mapping function constructed for the relatively compact domain Ω does not *assume*, in Ω, any value lying on the *boundary* of the conjectured image domain. This a priori assumption is justified only in the case of two boundary curves. Ahlfors noticed this difficulty, and had to overcome it by appealing to a version of the residue formula, where poles are permitted on the boundary of the domain; however the argument is sketchy. The original proof of Hilbert-Koebe is also given in the books Kodaira [6], Springer [9], Weyl [10]. A completely "non-classical" proof is due to Ahlfors-Bers [2].

(I am grateful to Prof. R. R. Simha who made me aware of these proofs from several papers that were published in German, when I was working on my Doctoral dissertation with Prof. S. Kumaresan).

In the following sections we recall some of the underlying theory and describe Koebe's General Uniformisation Theorem. At the end of the note we indicate how Uniformisation Theorem readily follows from it.

3.2 Riemann Surfaces and Uniformisation

A **Riemann surface** is a complex manifold of complex dimension 1. More specifically,

Definition 4. *A Riemann surface X is a connected Hausdorff topological space together with*

(i) *a system of complex coordinate charts $\Phi = \{\varphi_i : U_i \to V_i, i \in I\}$, where each φ_i is a homeomorphism from an open set U_i on to an open set $V_i = \varphi_i(U_i)$ in the complex plane $\mathbb{C}$, such that $\bigcup_{i \in I} U_i = X$, and*

(ii) *whenever $U_i \cap U_j \neq \emptyset$ the composition $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is biholomorphic.*

The charts $\varphi_i : U_i \to V_i, \varphi_j : U_j \to V_j$ satisfying Condition (ii) are called compatible and Φ is called a *complex atlas* on X.

Two complex atlases Φ and Φ' on X are called analytically equivalent if every chart of Φ is holomorphically compatible with every chart of Φ'. One can see that the notion of analytical equivalence is an equivalence relation.

An equivalence class of analytically equivalent atlases on X is called a complex structure on X.

For any given arbitrary atlas Φ in a complex structure, there exists a unique maximal atlas Φ^* consisting of all complex charts on X which are holomorphically compatible with every chart of Φ.

Thus a Riemann surface X is a connected Hausdorff topological space with a complex structure defined on it.

Unlike in the classical framework, the modern definition of Riemann surface does not require a defining function (in general a multivalued function). It is worth noting that the book of Hermann Weyl [10], "The Concept of a Riemann surface", published in 1913, opened a new era in the theory of Riemann surfaces. In the preface of the book it was mentioned that three events had a decisive influence on the form of his book:

"Brouwer's papers on topology, commencing 1909; the recent proofs of P. Koebe of the fundamental uniformisation theorems; and Hilbert's establishment of the foundation on which Riemann had built his structure, which was available for uniformisation theory, the Dirichlet principle."

Consider the polynomial $P[z,w] = z^2 + w^2 - 1 \in \mathbb{C}[z,w]$ which defines the "complex circle"

$$z^2 + w^2 = 1$$

in $\mathbb{C}^2$. Let $\Gamma = \{(z,w) \in \mathbb{C}^2 : P(z,w) = 0\}$. Then the problem is to find holomorphic functions f, g in a domain $\Omega \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that

1. $P(f(t), g(t)) = 0$ for all $t \in \Omega$
2. $(z,w) \in \Gamma \Rightarrow$ there exists $t \in \Omega$ such that $z = f(t), w = g(t)$.

It is holomorphically parametrized by $\mathbb{C}$ using $f(t) = \cos t$ and $g(t) = \sin t$. This would not be satisfactory because it is not a bijective parametrization, and also because these functions have essential singularity at ∞ whose behavior at ∞ is unclear.

The same circle can be bijectively parametrized by $U = \hat{\mathbb{C}} \setminus \{\pm i\}$,

$$f(t) = \frac{1-t^2}{1+t^2},\ g(t) = \frac{2t}{1+t^2}$$

which has poles $\pm i$. It can also be bijectively parametrized by

$$f(t) = \frac{1}{2}\left(t + \frac{1}{t}\right),\ g(t) = \frac{i}{2}\left(t - \frac{1}{t}\right),$$

for t varying in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and it can be compactified into $\bar{\mathbb{C}}$ by adding the two points 0 and ∞ which are poles in this case.

In 1882, Felix Klein discovered that every compact Riemann surface could be parametrized by a variable t, whose domain is in $\bar{\mathbb{C}}$. Poincaré was aware of this, and proved the more general form of Uniformisation theorem that the variable lies in a simply connected domain. For an interesting historical note in this respect see Abikoff [1].

A local coordinate chart on a Riemann surface X is also called a local uniformising parameter.

If $\pi : Y \to X$ is a topological covering, then Y too carries a Riemann surface structure such that π is a holomorphic local homeomorphism. Therefore, if we know the local parameters for Y, we can obtain the local parameters for X, via the composition with π. Therefore, if Y can be coordinatised using a single complex variable, that is if Y is biholomorphically equivalent to some domain of $\mathbb{C}$, then we can justify uniformising X by a single variable. We have to find such covering surfaces which are domains in $\mathbb{C}$. So, the uniformisation theorem asserts that if the universal covering surface of $X \neq \hat{\mathbb{C}}$, then X is biholomorphic to a simply connected domain in $\mathbb{C}$.

3.3 Planarity Condition

From the above examples it is clear that the converse of Theorem 3 is also true.

Suppose X is a compact planar Riemann surface. Then for any $p \in X, X^* = X \setminus \{p\}$ is non-compact and planar. If we have found a one-one holomorphic function f^* on X^*, then p cannot be an essential singularity for f, by Weierstrass' theorem, and it is clear that the extended holomorphic map $f : X \to \hat{\mathbb{C}}$ is one-one.

Therefore, it is enough to prove the General Uniformisation Theorem for non-compact Riemann surfaces.

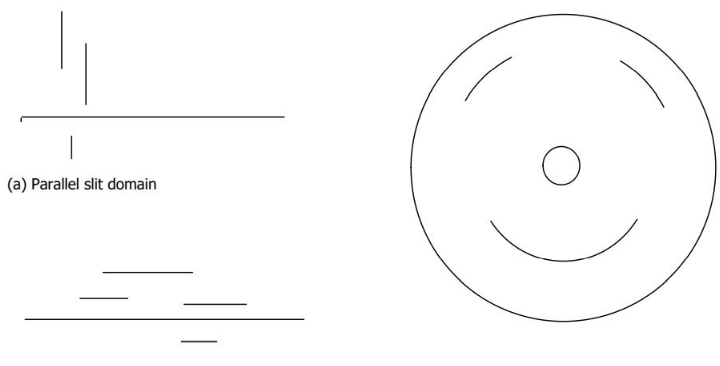
Theorem 5. *Let X be a non-compact planar Riemann surface, and $\Omega \subset X$ be a domain with compact closure and analytic boundary. Then Ω is biholomorphic to a domain in $\mathbb{C}$.*

The proof is generally based on the following idea: if Koebe's Theorem is true, then its proof for a relatively compact domain with analytic boundary in a Riemann surface must be the same as the proof of the theorem which says that a plane domain bounded by finitely many analytic Jordan curves is biholomorphic to a canonical domain in $\mathbb{C}$.

Therefore one approach in general is to show that a domain with good boundary on a planar Riemann surface is biholomorphic to a parallel slit domain or a domain with the circular ring with concentric circular arcs etc.

Definition 6. *(The parallel slit domain) This domain consists of the entire complex plane including the point at infinity, to which a number of parallel rectilinear slits have been applied. See Figures 1(a), 1(b).*

Definition 7. *(The circular ring with concentric circular slits) In the doubly connected case, this domain reduces to an annulus. For connectivity > 2, there are additional slits along circular arcs concentric with the circular ring. See Figure 1 (c)*



(a) Parallel slit domain

(b) Parallel slit domain

(c) Circular ring with concentric circular arcs

Figure : 1

There are a variety of proofs of this, for which the reader may refer Hemasundar [4], Kodaira [6], Simha [8], Springer [9].

As mentioned before we show that the uniformisation theorem follows easily from the Koebe's planarity theorem: Suppose X is a simply connected Riemann surface; then it can be mapped biholomorphically on to a horizontal slit region S; see Springer [9] . Since X is simply connected the boundary of slit region contains at most one boundary component. Therefore three cases can occur:

1. It may be the whole of the Riemann sphere.

2. If the boundary consists of a single point, that is if S is $S^2 \setminus \{p\}$, then it can be transformed into the whole finite plane by the mapping $1/(z-p)$. Clearly in this case it is the complex plane $\mathbb{C}$.

3. In the third case if S is the sphere with a slit of positive length, then we can map first on to a horizontal slit region with a slit of finite length on x - axis. Suppose it is mapped on to the

region with a cut along $[-1,\ 1]$ on x - axis. Then the mapping

$$\frac{1}{2}\left(z+\frac{1}{z}\right)$$

maps the region on to the interior of the unit disk which is our third region.
It is not hard to see now that the above mentioned three regions can not be mapped one on to the other region bilholomorphically.

Of course, there are several applications of Uniformisation Theorem, one can find in the books on Riemann surfaces. Here is a direct application of Koebe's General Uniformisation Theorem apart from deducing Uniformisation Theorem:
Application: In 1857 Riemann asserted that the complete family of inequivalent (biholomorphically distinct) complex structures on a compact base surface of genus $g \geq 2$ could be parametrized by $3g-3$ complex ($6g-6$ real) parameters which he named them moduli. The space of biholomorphically distinct complex structures on a given topological surface of genus g is known as moduli space. Moreover, all possible complex structures on the genus g base should be represented within the moduli space. This space is a complicated one and is a subject of much advanced research for higher dimensional complex manifolds. In the case when $g=0$, any compact Riemann surface is biholomorphic to Riemann sphere.

We can construct families of compact Riemann surfaces of a given genus by using Koebe's General Uniformisation Theorem. We can show explicitly in case of $g=1$, every compact Riemann surface belongs to this family. It is of some interest to know whether every compact Riemann surface of genus g belongs to this family. See Hemasundar [5].

We conclude with the words of Hermann Weyl commending Koebe in [10]

"To him above all we owe it that today the theory of uniformisation, which certainly may claim a central role in complex function theory, stands before us as a mathematical structure of particular harmony and grandeur".

Acknowledgment: I thank the referee for giving suggestions to improve this note to its best form.

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4. The Story of ICM 2010

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The first International Congress of Mathematicians (ICM) was neld in Zurich in 1897. Since then it has by and large been held regularly as a quedrennial event except for the world war years. The Congress has been organized since 1920 by the International Mathematical Union (IMU) jointly with local hosts from participating countries and has accquired great significance attracting participation from around the world on a large scale, and in turn the event has played an important role in mathematical developments over the last 120 years. For instance the 23 famous 'Hilbert problems' were announced by Hilbert in the Paris ICM of 1900. Many coveted prizes -the Fields medal (since 1936), the Nevanlinna Prize (since 1982 continued now as the IMU Abacus Prize), the Gauss Prize (since 2002) and the Chern medal (since 2010) are awarded at the opening ceremony of the Congress. For obvious historical reasons most of the Congresses have been held in the western world, though with increasing participation over the years from other regions, Asia in particular. The first ICM in Asia was at Kyoto in 1990 and the second one in Bejing in 2002. Inspired by these, there was a feeling that we should have one hosted in India, and it materialized, thanks largely to the leadership role played by Prof. M. S. Raghunathan, with the ICM of 2010 being held in India, at Hyderabad. The aim of this article is to recount some of the circumstances and interesting aspects relating to the event, as a tribute to Prof. Raghunathan, on the occasion of his 80th birthday. He was the Chairman of the Executive Organizing Committee for the ICM held in Hyderabad and if we are to remember his contribution to the development of mathematics in India, surely the story of the ICM must be told.

I may begin by recalling that aside from his role in its planning and execution, the 2010 ICM at Hyderabad was very much the brainchild of Professor Raghunathan. He had been a member of the Executive Committee of the International Mathematical Union during the 2003 - 06 term, immediately following the ICM in China and felt that the mathematical community in India was large enough and mature enough to justify the holding of an ICM in the country. He shared his ideas with several mathematicians. One question that bothered many was: would the considerable expenditure involved in hosting an event, which in recent years had been attracting 3000 plus delegates, be justified in terms of the benefits that may be expected to accrue, in the overall context of the country. The consensus was that if we could ensure broad participation of the country's mathematical community it would expose them to the current status of top quality mathematics from around the world and if we could ensure the participation of bright young minds in the country with some penchant for mathematics, it was possible that some of them may be inspired by the Field Medallists who they will witness receiving their prize just after the inauguration. As to the cost Prof. Raghunathan had arrived at an estimate of Rs.12 crores as the funds needed for the event. It was worth a try.

We wanted to bid for the event in 2006 but the IMU was reluctant to hold two consecutive Congresses in Asia. It was therefore decided to make a bid for the ICM in 2010 and so with the approval of the DAE (Department of Atomic Energy, Government of India) and the assistance of the then President of the IMU, Sir John Ball, we made a bid for 2010. A presentation had to be made in support of the bid at the IMU General Assembly to be held in Madrid just before the ICM of 2006, where the final decision for the venue for ICM 2010 would be made, between possible contenders. Prof. Raghunathan spearheaded the task through the NBHM (National Board for Higher Mathematics, Government of India) with threadbare discussion in the Board on various details, to present a strong case, anticipating all the doubts that could be raised in the matter. And indeed we won the bid, though there was another contending bid involved almost through to the end.

The venue for ICM 2010 suggested in our original bid was the exhibition ground in New Delhi; however the IMU in giving its approval (after inspection of the venue) had expressed some reservations about the infrastructure available at the venue. We envisaged some construction work to fill in the infrastructural gaps, but uncertainties remained. With the approval in hand, Prof.

Raghunathan called a meeting of mathematicians from all parts of the country for a brainstorming session in Delhi. The purpose was to evolve a broad consensus on how the ICM should be organized and to discuss the financial aspects involved. I was one of the people called for the meeting. Whilst waiting at Hyderabad airport I noticed a large hoarding about the Hyderabad International Convention Centre (HICC). I mentioned this to Prof. Raghunathan at the meeting and he asked me to visit the place which I did immediately on return to Hyderabad. Whilst the convention centre was not nearly as large as the one which had hosted the ICM in Beijing in 2002, it seemed to have enough space for many parallel sessions to go on simultaneously and a flexibly sized hall that could seat more than 5000 delegates. A few days later Prof. Raghunathan visited Hyderabad together with a group of mathematicians to see the convention centre, following which he wrote to Prof. Ball proposing the HICC as an alternative venue. Prof Ball came to Hyderabad and approved the HICC. And so the ICM was shifted to Hyderabad.

Professor Raghunathan recommended to the DAE that an executive organizing committee (EOC) consisting of thirteen members including 11 mathematicians and one representative each from the DAE and DST be appointed. I had the privilege of being appointed the Secretary to the EOC. The EOC was to oversee both the organizational and financial aspects of the conference. The EOC met twice in the first year after it was formed, quarterly in the second year then bimonthly and finally every month in the last year preceeding the conference. Prof. Dinesh Singh, then Pro Vice Chancellor of Delhi university was inducted into the committee in March 2010, just 5 months before the conference, a decision that Prof. Raghunathen felt should have been taken much earlier. We realized that it was necessary to have a person with some influence with the Delhi bureaucracy. Prof. Raghunathan had tried to persuade the IMU to call the Prime Minister to preside over the inauguration of the Congress. This was not only because he happened to be a respected former academic, but also because in India ever since independence the Prime Minister had been in charge of science policy. The IMU however insisted that a tradition had been established that the head of state of the host country preside over the inauguration, and that this tradition must be continued. Prof. Dinesh Singh was of invaluable help in persuading a reluctant President to agree to preside over the inaugural event.

One matter of great concern to Prof. Raghunathan was the risk of terrorism. People still remembered the terrible terrorist acts of the 11th of September 2001 and Mumbai had been shaken to its core by events on the 26th of November 2008. Last but not least the scientific community had been hit by an act of terrorism in December 2005, at the Indian Institute of Science, Bangalore. If not Israeli, there would certainly be jewish delegates at the Congress. It was an international event and terrorists seek international attention. The organizers were in touch with the police on this matter. We finally decided to keep everything low key and not have any armed security presence at the venue.

Almost all meetings of the EOC were chaired by Prof. Raghunathan. Decisions were taken in the EOC by consensus and implemented by the local (Hyderabad) members of the committee, or the specific members of the EOC concerned with the task involved. The General Body of IMU traditionally meets just before the ICM in the host country of the Congress, usually at a different location. There was a 2-day meeting of the General Body of the IMU just prior to the ICM. This 2-day meeting was held at the Indian Institute of Science (IISc), Bangalore, and was organized by the Bangalore members of the committee. Remarkably with 12 to 13 heads meeting so often over a period of 4 to 5 years I cannot remember even a single instance of any friction amongst members of the EOC. There were debates and decisions were taken by consensus. Prof. Raghunathan's cool head, fair mindedness and attempt to involve all members of the committee in the implementation of some or other aspect of the conference ensured that the ICM engine ran smoothly. We also appointed a PCO (Professional Conference Organizer), M/s K W Conferences. They were extremely useful and helped us to save more than the fee we paid them.

Needless to say all the talks at the ICM - which are organized by the IMU through appointment of international expert committees - went smoothly, and so did the various ancilliary events. Any account of the ICM would be incomplete without a mention of the cultural events at the conference.

The biggest of the cultural events were two performances of the play A Disappearing Number by the London based theatre group Complicité. The play was on the relationship between Ramanujan and Hardy. It had received the Critics' Circle Theatre award, the Lawrence Olivier award, been performed in Europe, Australia and the USA but not yet come to India. Ms. Sanjana Kapoor of Prithvi Theatre (Mumbai) fame had attempted to bring the play to India in 2008/9 but had failed to get enough sponsors. It was very expensive to get the play because besides the travel fare for all those involved in the production of the play one also needed to pay for the transportation of 20 tonnes of cargo. It was a technology-driven play and required very special type of theatre props and settings for its performance. Prof. Marcus du Sautoy of Cambridge University wrote to Prof. Raghunathan urging him to make it possible for the play to be performed in India. There were doubters on the EOC. Would it be right to spend public funds on such an "elite" and "restricted" activity. It was immediately decided that if we did somehow manage to bring the play to India no public funds would be used for the purpose. Fortunately there were some generous donors who had given money for the Congress. Finally Profesor Raghunathan took the plunge to attempt to bring the play to India. We contacted Ms Kapoor and it was agreed to share the costs with Prithvi Theatre. We hoped to bring down costs further by finding sponsors in Chennai, Kolkata and Delhi, cities with known traditions of theatre, but unfortunately they could not be found. It was a nightmare to find a theatre that would agree to make the necessary adjustments required to install all the equipment that the play needed and in fact we had at one stage dropped the idea of having the play until a miracle happened and the Brahmakumaris assented to the requisite restructuring in their theatre (at our cost, of course).

The second prominent event was Vishwanathan Anand playing simultaneous chess with 40 mathematicians. This was the only event of the ICM that the press showed any interest in. In 2006 the Indian ambassador in Madrid had been extremely kind to invite for dinner the entire Indian delegation that attended the ICM. There we met Anand and his wife Aruna. It is there, I think, that the idea came to Prof. Raghunathan's mind. Anand graciously agreed. This event drew considerable interest, both of the public and of the delegates - Anand won all but one match - he drew with a 14 year old budding mathematician from Bangalore. Immediately after the event the press requested a press conference which Prof. Raghunathan addressed. The first question was the inevitable "What do you think of Vedic Mathematics". Prof. Raghunathan sent shock waves through the press corp by initially saying something derisory but then clarified that what he meant was that Vedic Mathematics was mainly concerned with algorithms and they would now be of greater interest to computer scientists - mathematics was more concerned with 'proof'. This however was not the end of the Anand story. Since Prof. Mumford and Anand would be present in Hyderabad at the time it was suggested by us that the University of Hyderabad give them both an honorary degree. Prof. Hasnain, the Vice Chancellor of the University (a great help and supporter of the ICM) readily agreed and proposals were sent to the Ministry of External Affairs and the MHRD for permission. Some bureaucrat in the MHRD got it into his head that Anand had taken Spanish citizenship and insisted that we send our proposal to the MEA instead. No amount of persuasion would budge him. We were all outraged. Prof. Raghunathan was clear that this matter must be taken to the press. The media devoured the news and it was all over national TV. It caused much embarrassment to the MHRD and the minister, Mr. Kapil Sibal personally apologised. To complete the story the Anand's were very cool about the whole matter and Mrs Anand just sent me by email a copy of Anand's Indian passport.

As may be expected, we had musical performances at the Congress - one in Carnatic and one Hindustani. But Prof. Raghunathan had two other ideas. He suggested that we invite Prof. Sunil Mukhi of the TIFR (Tata Institute of Fundamental Research, Mumbai) to give two lectures on appreciation of Hindustani classical music and invite Simon Singh to describe how and why he wrote the book Fermat's Last Theorem. All three lectures were very well attended.

There is one other matter which saw Prof. Raghunathan make a quick decision, thinking on his feet. The recipient of the Chern Medal at the ICM was Prof. Louis Nirenberg. He was 85, not very steady on his feet and needed the support of a walking stick to move around. The stage at the inauguration was rather high and we were sure that he would not be able to climb the steps to receive the medal from the President. The President's security had given the organizers permission to seat 9 people on the dais. Since the prize winners were only revealed on the morning of the inauguration itself, this was not a problem we could have anticipated. The security were rather rigid about this and we knew that if we ask them at the last minute to make an exception and allow Prof. Nirenberg to sit on the dais too, they would decline us permission. Luckily for us the President's husband who had been invited by us on Rashtrapati Bhavan's request was indisposed and did not come. Prof. Raghunathan immediately decided that we would seat Prof. Nirenberg on the dais. There would be 9 people on the dais and the security would be none the wiser.

So were the aims of Prof. Raghunathan fulfilled. Fifteen hundred Indian mathematicians attended the ICM, a thousand funded directly by the EOC and 500 by the DST. About a 100 carefully selected senior school students participated as volunteers. There were two 'public lectures' attended by over 2500 school students - one on The Proof by Gunter Ziegler and the other by Bill Barton titled 'Where is mathematics taking me, an exciting ride into the future'. The jury is out. We tried.

It was surely a privelege for me to work with Professor Raghunathan.

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5. A Review on a Book

A Course in Calculus and Real Analysis (Second Edition),
By Sudhir R. Ghorpade and Balmohan V. Limaye (Springer, 2018)

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Calculus of a single variable is a basic course which is in the curriculum of most of the undergraduate students in science and engineering. This is also one of the fundamental courses for the undergraduate students in mathematics. Prior to entering an undergraduate program, these students have already been introduced to most of the standard topics in calculus of a single variable, such as limit, continuity, differentiation and integration in their high school curriculum. They have also been exposed to some of the applications of differentiation and integration. Sequences and series of real numbers are also introduced at this level to some extent. For instance, exponential functions are defined in terms of power series in high school. However, at the school level, the concepts mentioned above are neither introduced nor defined in a mathematically rigorous manner. Further, most of the associated results are also not proved at this stage.

Calculus at the undergraduate level commences as a continuation of the calculus taught at high school. However, the manner in which the subject is approached and the syllabus can vary vastly as they depend on the curriculum and the instructors teaching the course. Catering to a variety of needs, there are a large number of books already available on calculus. Some books assume the concepts and the results taught at the school level and principally deal with the applications of the concepts. Of course, the applications of calculus are important for scientists and engineers. But, a significant component of education is the training of the mind for precise and rigorous thinking. To achieve this, it is imperative that the student is exposed to the basic results and their proofs. Further, a large number of undergraduate students are also desirous of knowing the genesis of concepts, proofs and their applications. Hence, many authors of the books on calculus attempt to maintain a balance between mathematical rigour and applications. For instance, some books present the statement of the Bolzano-Weierstrass theorem without a proof but using this result, they develop the requisite analysis for calculus. Some present the completeness property of the real number system as an axiom and prove even the Bolzano-Weierstrass theorem. The same books may avoid the proofs of some important results in calculus such as the L'Hospital rule, Pappus theorems, convergence of Newton-Raphson method etc.

We rarely come across a book on calculus of a single variable which is all inclusive and presents the proofs of all the statements presented in the book. The book under the review is one such book. This book is not only self contained, but its approach is also mathematically rigorous. The book even outlines the construction of real number system in the Appendix. This book does not even assume the area of the circle for measuring the angles in terms of radians and also for defining the trigonometric functions.

The topics in a standard course of calculus of a single variable are covered in the first nine chapters in this book. The tenth chapter discusses sequences and series of realvalued functions of a real variable which is typically not addressed in a standard text on calculus. Concepts and results are explained and illustrated with numerous examples and geometric figures. The book contains a large number of problems. Geometric interpretations of various results, concepts and methods are provided wherever possible. This will enable the student to visualize the ideas. All the steps of the proofs and illustrated examples are explained very clearly. The exposition presented in the "Notes and comments" can be inspiring for students.

The presentation of the topics covered in this book is unusual in comparison to a standard textbook on calculus. For instance, the trigonometric functions are introduced in Chapter 7. Examples using these functions are also avoided in the preceding chapters, where the concepts of continuity and differentiability are discussed. Trigonometric functions are typically used in many standard examples and counterexamples, when the concepts of limit, continuity and differentiability are discussed. However, the reasons for avoiding the trigonometric functions till Chapter 7 are explained. In a standard textbook of calculus, the geometric properties such as the intermediate value property, monotonicity, convexity and local extrema are discussed after the introduction of the concepts of either continuity or differentiability. This book does this differently.

When the undergraduate students are taught calculus in a mathematically rigorous manner, they will encounter the proofs of the results, perhaps already known to them, and the logical reasoning behind them for the first time. At this juncture, it is natural for mathematically inclined students to ask a lot of questions. Answers for many such questions can be found in this book. Moreover, the book has many interesting results in every topic, which are usually not found in standard text books. Hence this book can be an excellent reference book for the instructors, mathematically inclined students who are doing a course on calculus and undergraduate students in mathematics. This book can also be used by students who have already done a course on calculus or real analysis for consolidating their knowledge. However, it is difficult to adopt this book as a text book for introductory courses in either calculus or real analysis.

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No matter how correct a mathematical theorem may appear to be, one ought never to be satisfied that there was not something imperfect about it until it also gives the impression of being beautiful. — George Boole — Quoted in D MacHale, Comic Sections (Dublin 1993).
Source: *https://mathshistory.st-andrews.ac.uk/Biographies/Boole/quotations/*

If one has really technically penetrated a subject, things that previously seemed in complete contrast, might be purely mathematical transformations of each other. – John Von Neumann
Source: *https://en.wikiquote.org/wiki/John_von_Neumann*

6. What is happening in the Mathematical world?

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6.1 Decades-old Classification Problem of torsion-free abelian groups is solved

Practitioners in the field of descriptive set theory, have a way of rating the difficulty level of classification problems - sometimes concluding that a given classification task is relatively easy to carry out, and sometimes discovering that it's too hard.

For decades, one classification problem - involving a particular set of infinitely large objects consisting of torsion-free abelian groups (or TFABs) - obstructed researchers. This problem was first raised in 1989 by the mathematicians Harvey Friedman and Lee Stanley in a paper that, according to Paolini, "introduced a new way of comparing the difficulties of classification problems for countable structures, indicating that some things are more complicated than others."

Gianluca Paolini (Left) of the University of Turin and Saharon Shelah (Right) of the Hebrew University of Jerusalem have answered the decades-old question of just how difficult it is to classify torsion-free abelian groups. Their strategy displays an incredible amount of cleverness in transforming a complicated problem into something easier one.

In their 1989 paper, mathematicians Friedman and Stanley showed that answering the question whether two countable graphs are isomorphic is maximally complicated. In technical terms, the family of all countable graphs is "Borel complete." In the years since, Friedman, Stanley and others have identified several classes of mathematical objects that satisfy the criteria for Borel completeness, including trees - a special class of graphs- and linear orders.

While many different cases were considered in the 1989 paper, one instance- concerning the aforementioned - the class of torsion-free abelian groups - resisted classification by isomorphism. Paolini and Shelah finally found a way to break through, earlier this year.

This is achieved by constructing a Borel map f associating to each graph a torsion-free abelian group such that two countable graphs, G and H, are isomorphic to each other if and only if f(G) and f(H) are isomorphic to each other. The task is performed using the notion of a universal graph, one whose subgraphs account for all graphs.

Sources: *https://www.quantamagazine.org/mathematicians-solve-decades-old- classification- prob lem-20210805/*

6.2 A notable advance in computer assisted proofs

Proof-assistant software handles an abstract concept at the cutting edge of research, revealing a bigger role for software in mathematics. Peter Scholze, a Fields medalist of 2018 and one with a track record of introducing revolutionary concepts, wants to rebuild much of modern mathematics, starting from one of its cornerstones. Now, he has received validation for a proof at the heart of his quest from an unlikely source: a computer.

Mathematicians have long used computers to do numerical calculations or manipulate complex formulas. But systems known as proof assistants go deeper. The user enters statements into the system to teach it the definition of a mathematical concept - an object - based on simpler objects that the machine already knows about. A statement can also just refer to known objects, and the proof assistant will answer whether the fact is 'obviously' true or false on the basis of its current knowledge. If the answer is not obvious, the user has to enter more details. Proof assistants thus force the user to lay out the logic of their arguments in a rigorous way, and they fill in simpler steps that human mathematicians had consciously or unconsciously skipped. Once researchers have done the hard work of translating a set of mathematical concepts into a proof assistant, the program generates a library of computer code that can be built on by other researchers and used to define higher-level mathematical objects. In this way, proof assistants can help to verify mathematical proofs that would otherwise be time-consuming and difficult, perhaps even practically impossible, for a human to check.

While machine may not take over the creative aspects of mathematics any time soon, they could well play an increasingly important role in research - and this particular feat could be a turning point towards its acceptance.

Scholze set forth the ambitious plan - which he co-created with his collaborator Dustin Clausen from the University of Copenhagen - in a series of lectures in 2019 at the University of Bonn, Germany, where he is based. The two researchers dubbed it 'condensed mathematics', and they say it promises to bring new insights and connections between fields ranging from geometry to number theory.

If Scholze and Clausen's vision is realized, the way mathematics is taught to graduate students in 50 years' time could be very different than it is today. There are a lot of areas of mathematics that in the future likely to be affected by his ideas.

Until now, much of that vision rested on a technical proof so involved that even Scholze and Clausen could not be sure it was correct. But in June 2021, Scholze announced that a project to check the heart of the proof using specialized computer software had been successful.

Scholze laid out his challenge to proof-assistant experts in December 2020, and it was taken up by a group of volunteers led by Johan Commelin, a mathematician at the University of Freiburg in Germany.

On 5 June 2021 - less than six months later - Scholze posted that the main part of the experiment had succeeded. The crucial point of condensed mathematics, according to Scholze and Clausen, is to redefine the concept of topology, one of the cornerstones of modern mathematics.

Around 2018, Scholze and Clausen began to realize that the conventional approach to the concept of topology led to incompatibilities between three mathematical universes - geometry, functional analysis and p-adic number - but that alternative foundations could bridge those gaps. Many results in each of those fields seem to have analogues in the others, even though they apparently deal with completely different concepts. But once topology is defined in the 'correct' way, the analogies between the theories are revealed to be instances of the same 'condensed mathematics', the two researchers proposed. It is a kind of grand unification of the three fields.

Sources: *https://www.nature.com/articles/d41586-021-01627-2*

6.3 Swiss researchers calculate pi to new record of 62.8 trillion figures

Knowing pi to a suitable degree of approximation is incredibly important because it appears everywhere, from the general relativity of Einstein to corrections in your GPS to all sorts of engineering problems involving electronics. From theancient Babylonian times, humans have been trying to approximate the constant. Swiss researchers have now calculated itto a new world-record level of precision, hitting 62.8 trillion figures using a supercomputer. The previous world-record pi calculation achieved 50 trillion figures. The calculation took 108 days and nine hours. Its efforts were almost twice as fast as the record Google set using its cloud in 2019, and 3.5 times as fast as the previous world record in 2020.

Sources:

1. *https://www.sciencealert.com/scientists-invented-a-machine-that-generates-mathematics-we-ve-never-seen-before*
2. *https://www.theguardian.com/science/2021/aug/17/new-mathematical-record-whats- the -point-of-calculating-pi*

6.4 Israeli researchers give a statistical solution to the 3-body problem

Researchers from Haifa's Technion published a statistical solution to the 'three-body problem,' using the theory of random walks, which models a "drunkard's movement."

Isaac Newton was the first to formulate mathematical principles that made it possible to accurately predict the motion of two massive celestial bodies in close proximity to each other. But Newton soon discovered that when another body was added to the system, he failed to find an accurate general solution. The "three-body problem" has remained without a mathematical solution for some 200 years, despite the best efforts of scientists.

In his work on the problem, published in the Royal Mathematical Journal in 1889, for which he was awarded a gold medal and 2,500 Swedish kronor, the French mathematician Henri Poincaré, showed that the interactions between the three bodies are fundamentally chaotic and there is no formula to describe it. This proof is considered one of the foundations of chaos theory. The absence of a deterministic solution to the "three-body problem" means that scientists cannot predict what happens during a close interaction between two orbiting bodies such as the Earth and the Moon, and a third object approaching them.

Prof. Hagai Perets(left) and his doctoral student Yonadav Barry Ginat(right) from the Technion - Israel Institute of Technology, Haifa, have now described a statistical solution to the problem. While it is not possible to predict the actual result of each three-body interaction over a period of time, Ginat and Perets describe the probability of various possible outcomes, at any given time.

The method Perets and Giant used is called the "density-based method," used by researchers since the mid-70s. This method is based on calculating the density of states in the phase space (the space of configurations of all three bodies and their velocities).

Sources:

1. *https://www.haaretz.com/israel-news/.premium.MAGAZINE-israeli-researchers-just-solved-problem-baffling-physicists-since-the-days-of-newton-1.10125057*
2. *https://scitechdaily.com/a-centuries-old-physics-mystery-solved/*

6.5 Spacetime Crystals: A new formula may solve an old problem in understanding of the Universe

A Penn State scientist studying crystal structures has developed a new mathematical formula that may solve a decades-old problem in understanding spacetime, the fabric of the universe proposed in Einstein's theories of relativity.

For calculations to work within relativity, scientists must insert a negative sign on time values that they do not have to place on space values. Physicists have learned to work with the negative values, but it means that spacetime cannot be dealt with using traditional Euclidean geometry and instead must be viewed with the more complex hyperbolic geometry.

Now, Prof. Venkatraman Gopalan, professor of materials science and engineering and physics at Penn State University developed a two-step mathematical approach that allows the differences between space and time to be blurred, removing the negative sign problem and serving as a bridge between the two geometries.

For more than 100 years, there has been an effort to put space and time on the same footing, but that has really not happened because of this minus sign. This research removes that problem at least in special relativity. Space and time are truly on the same footing in this work. Gopalan's idea of general relativistic spacetime crystals and how to obtain them is both powerful and broad.

In addition to providing a new approach to relate spacetime to traditional geometry, the research has implications for developing new structures with exotic properties, known as spacetime crystals.

Gopalan's method involves blending two separate observations of the same event. Blending occurs when two observers exchange time coordinates but keep their own space coordinates. With an additional mathematical step called renormalization, this leads to "renormalized blended spacetime."

Sources: *https://scitechdaily.com/spacetime-crystals-new-mathematical-formula-may-solve - old-problem-in-understanding-the-fabric-of-the-universe/*

6.6 Analysis of an ancient Babylonian tablet reveals use of geometry in civil works about 3700 years ago

A mathematician from the University of New South Wales has found the oldest known example of applied geometry, on a 3,700-years-old small piece of Babylonian clay tablet dug up in central Iraq. The circular tablet Si.427 features a diagram drawn by a Babylonian surveyor about 1900-1600 BCE whose analysis now indicates that applied mathematics was used to survey land and measure its boundaries with extreme precision.

The new revelation was done by Daniel F Mansfield of the School of Mathematics and Statistics at the University of New South Wales, who studied one of the most complete examples of applied geometry from the ancient world - Si.427. The tablet was discovered in 1894 by French archaeologist Vincent Scheil at Sippar, the site Near Eastern Sumerian and later Babylonian city in today's Iraq, and is currently in a museum in Istambul. It is generally believed that trigonometry was developed by the ancient Greeks studying the night sky in the second century BC. But this tablet revealed that the Babylonians had developed their own alternative 'proto-trigonometry' to solve problems related to measuring the ground, not the sky.

Four years ago Mansfield had researched together with Norman Wildberger on another old Babylonian tablet known by the name Plimpton 322, consisting of a table involving 15 Pythagorean triples, and had concluded that the table may have been used by ancient mathematical scribes to construct palaces and temples, build canals, or survey land. Further pursuit of the idea inspired the current research.

Mansfield published a paper in the journal Foundations of Science where he mentioned that Si.427 was used to survey a piece of land for the purpose of selling. The tablet details a muddy field with various structures, including a tower, built upon it. He also provides an analysis of a table of rectangles with information about which sides are regular and which sides are not, indicating that there was some contemporary interest in rectangles with regular sides in Mesopotamian mathematics. The tablet shows that unlike earlier field plans found from the period, these measurements were made with unusually high precision.

The tablet is engraved with three sets of Pythagorean triples: three whole numbers for which the sum of the squares of the first two equals the square of the third. The triples engraved on Si.427 are 3, 4, 5; 8, 15, 17; and 5, 12, 13. These were likely used to help determine the land's boundaries.

Though the tablet does not express the Pythagorean theorem in the familiar algebraic form it is expressed today, coming up with those triples would have required understanding the general principle that governs the relationship between length of the sides and the hypotenuse. The Pythagoras principles were a watershed moment in mathematics that was defined in the 6^{th}century BC, over 1,000 years after the markings on the Si.427 tablet. Nobody expected that the Babylonians were using Pythagorean triples in this way. What is surprising to Mansfield, however, is the level of theoretical sophistication the tablet reveals the ancient Babylonians to have had at such an early stage of human history.

Sources:

1. *https://www.indiatoday.in/science/story/babylonians-knew-about-pythagoras-theorem - 1 -000-years-before-it-was-given-by-pythagoras-1837671-2021-08-06*

2. *https://www.livescience.com/earliest-form-of-pythagorean-triplet*
3. *https://www.maths.unsw.edu.au/news/2021-08/daniel-mansfield-Si427*

6.7 Awards

6.7.1 The 2020 Wolf prizes awarded to Simon Donaldson and YakovEliashberg

Professor Sir Simon Donaldson (top), Chair in Pure Mathematics at Imperial College, London and Professor at Stony Brook University in New York and professor Yakov Eliashberg (bottom), the Herald L. and Caroline L. Ritch Professor in the School of Humanities and Sciences, at Stanford University, have been jointly awarded the 2020 Wolf Prize for Mathematics, one of the most prestigious academic accolades, for their contribution to differential geometry and topology. They share between them the prize money of $1,00,000.

Sir Simon has given revelatory insights into the fourth dimension, complementing our three dimensional perception of the world. While it is common to use mathematics to solve physics problems, Sir Simon applies physics to solve mathematical problems - an approach that has ultimately led to impressive advances in both.

Professor Donaldson is renowned for his work on the topology of smooth (differentiable) four-dimensional manifolds and the Donaldson-Thomas theory. His research includes a unique combination of novel ideas in global non-linear analysis, topology, algebraic geometry, and theoretical physics. As a graduate student at the University of Oxford, Professor Donaldson made a discovery that earned him international esteem and stunned the mathematical world. He showed that there are phenomena in 4-dimensions which have no counterpart in any other dimension. This went against the accepted understanding at the time.

Prior to this, Professor Donaldson had been awarded the Fields Medal in 1986, and the $3m Breakthrough prize in 2014.

Eliashberg is one of the founders of symplectic and contact topology, a field that arose in part from the study of various classical phenomena in physics that involve the evolution of mechanical systems, such as springs and planetary systems. "The emergence of symplectic and contact topology has been one of the most striking long-term advances in mathematical research over the past four decades," the Wolf Foundation said in a statement.

Eliashberg has been a professor at Stanford since 1989 and has helped 35 graduate students earn their PhDs. He says that interacting with students has been one of the most rewarding aspects of his position. "It takes a lot of time but it is extremely uplifting and also very helpful to my own work," he said.

The Wolf Prize is awarded in Israel each year to outstanding scientists and artists from around the world for "achievements in the interest of mankind and friendly relations among peoples." Along with the Fields Medal and Abel Prize, it is considered the closest equivalent to a Nobel Prize in mathematics.

Sources:

1. *http://www.imperial.ac.uk/news/225233/celebrated-imperial-mathematician- lauded -wolf-prize/*
2. *https://news.stanford.edu/2020/01/17/yakov-eliashberg-awarded-wolf-prize-mathematics/*

6.7.2 Shanti Swarup Bhatnagar Prizes of 2021 awarded

The 2021 awards of the Shanti Swarup Bhatnagar Prize for science and technology, the country's highest science award,were announced during the 80th foundation day of the Council for Scientific and Industrial Research (CSIR) on Sunday. The award is named after the founder Director of the Council of Scientific & Industrial Research (CSIR), the late Dr (Sir) Shanti Swarup Bhatnagar.

Bhatnagar Awards, each of the value of Rupees five lakh, are awarded annually for notable and outstanding research, applied or fundamental, to Indian scientists below the age of 45 in seven fields-Biology, Chemistry, Environment Science, Engineering, Mathematics, Medicine and Physics.

In the Mathematical sciences category, Dr Anish Ghosh (left), School of Mathematics, TIFR, Mumbai, and Dr Saket Saurabh (right), The Institute of Mathematical Sciences, Chennai, were announced winners.

Anish Ghosh works at the interface of Ergodic theory, Lie groups, and Number Theory. Ergodic theory offers a mathematical means to study the long-term average behavior of various complex dynamical systems. The work of Ghosh specializes in harnessing the understanding of these behavior patterns dealing with problems in Number theory. He has earlier been recipient of the Swarna Jayanti Fellowship of DST and the B. M. Birla Science Prize, and is elected Fellow of the Indian Academy of Sciences.

Saket Saurabh (right), works in Theoretical Computer Science, specializing in parameterized complexity, exponential algorithm, group algorithms and game theory. He has also been a recipient of the 2020 ACM India Early Career Researcher award, and is elected Fellow of Indian Academy of Sciences.

Sources: *https://www.indiascinceandtechnology.gov.in>listinpage*

6.7.3 Indian students bag 5 medals in International Mathematical Olympiad 2021

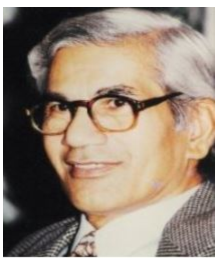
Indian team secured five medals (1 gold, 1 silver and 3 bronze) in the International Mathematical Olympiad (IMO) 2021, held in Russia. Bengaluru student Pranjal Srivastava secured the gold medal. Pranjal is the first Indian student to score two gold medals in IMO. Anish Kulkarni from Pune won the silver medal. Ananya Ranade, Pune; Rohan Goyal, New Delhi; and Suchir Kaustav, Ghaziabad bagged bronze medals. The Olympiad was held online. India secured 26th position in the competition. Earlier in 2001 and 1998 India could reach to the 7th position.

The selection of the team for IMO 2021 was carried out through a two-stage procedure. The first stage consisted of a nationwide examination namely the Indian Olympiad Qualifier in Mathematics (IOQM) in which over 17 thousand students appeared. This examination was conducted by the Mathematical Teachers' Association in nearly 175 centres across the country. And the second stage, the Indian National Mathematical Olympiad (INMO) examination, held for 1266 students, was conducted by HBCSE at 25 centres across the country.

Sources: *https://english.mathrubhumi.com/technology/science/indian-students-bag-5-medals -in-international-mathematical-olympiad-2021-1.5854619*

6.8 PU math Professor IBS Passi passes away at the age of 82

Professor Inder Bir Singh Passi, former Dean of university instructions (DUI) and professor emeritus of the department of mathematics at Panjab University, passed away on 2nd October, 2021 at the age of 82. Vice-Chancellor Prof. Raj Kumar condoled his demise saying: "With his passing away, PU has lost a stalwart. Passi was an academic par excellence."



Prof. Passi was a noted group-theorist, and has made significant contributions to certain aspects of theory of groups, especially to the study of group rings. His results on the dimension subgroups, augmentation powers in group rings, and related problems have received wide recognition. His 1979 monograph summarizing the state of the subject is a basic reference source.

After his superannuation from the Panjab University he continued to be active in research and teaching, with affiliations to the Harish-Chandra Research Institute, Allahabad (2000 - 05), the Indian Institute of Science Education and Research, Mohali (2007 –), and the Ashoka University, Sonipat.

His numerous distinctions include the award, in 1983, of the Shanti Swarup Bhatnagar Prize for Mathematical Sciences, the Meghnad Saha award of UGC for Research in Theoretical Sciences, and the Prashant Chandra Mahalnobis Medal of the Indian National Science Academy. He was elected Fellow of the Indian Academy of Sciences, the National Academy of Sciences and the Indian National Science Academy, and served on the Council of the Indian National Science Academy during 2015 - 17. He was President of the Indian Mathematical Society (2006/07), and President of the Mathematics Section of the Indian Science Congress Association (1998/99). He served as Editor of the Journal of the Indian Mathematical Society (1985 - 91), the Journal of Group Theory (1998 - 2001), and the Panjab University Research Bulletin (1995/96).

Professor Passi has played an important role in raising the standards of mathematics in India. The Mathematics Consortium accorded special felicitations to him at its international conference at the Banaras Hindu University, Varanasi, in 2016. Professor Passi is survived by his wife and two daughters.

Sources:

1. *https://en.wikipedia.org › wiki › Inder_Bir_Singh_Passi*
2. *https://www.hindustantimes.com › chandigarh-news › p...*

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7. TMC Activities (January-July 2021)

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7.1 A one-day National workshop on "Modern Embedded Systems and Computing: Challenges and Opportunities"

This workshop was conducted on 19th Feb. 2021, by Indian Academy of Industrial and Applicable Mathematics (IAIAM) jointly with TMC and the Quantum University, Roorkee, Uttarakhanda, in the online form. The participants were students/research scholars and faculty members from the Quantum University and nearby institutes. Total 300 participants were in the seminar hall watching the sessions in addition to 75 participants on the online meet.

Prof. Ramesh Gaonkar, a well-known author of books on microcontrollers, as well as Professor Emeritus from Syracuse University, US, was the keynote speaker. Dr. Vivek Kumar, the Vice Chancellor of Quantum Univ. presented the inaugural address. There were four 1-hour talks as follows: Prof. Ramesh Gaonkar (Recent Changes in Technology and its impact on Engineering Curricula), Prof. V. D. Pathak (Quantum Computing), Prof. Ajay V. Deshmukh (Role of Embedded Devices in Industry 4.0, and Computational Challenges in Industrial Embedded Systems), Prof. S. A. Katre (Aspects of Cryptography in Cyber Security Domain). Prof. Gulshan Chauhan, Dean Faulty of Technology, Quantum Univ. presented a vote of thanks. Dr. Aditi Sharma (Quantum U.) and Dr. Ajay Deshmukh (IAIAM) coordinated the workshop.

7.2 The NASI-TMC 3-week Summer School on Differential Geometry

This Summer School, funded by the National Academy of Sciences, India (NASI) and The Mathematics Consortium (TMC), was organized online as a part of "Azadi ka Amrit Mahotsav" celebrations by the Department of Mathematics and Statistics, Central University of Punjab (CUP), Bathinda, during July 05-24, 2021. Prof. Gauree Shanker, Head, Department of Mathematics and Statistics, CUP, worked as the Convener. Dr. Krishnendu Gongopadhyay, Head, Department of Mathematical Sciences, IISER, Mohali was the Co-convener. This summer school was the 3rd activity arranged under NASI-TMC programmes. The Summer School received 683 applications across the country and abroad, out of which 216 applicants were shortlisted and 134 actively participated in all the sessions.

Inaugural Session was conducted on July 05, 2021 at 9:30 AM in presence of Prof. Gauree Shanker (Convener), Dr. Krishnedu Gongopadhyay (CoConvener), Prof. Satya Deo (Guest of Honour), General Secretary, NASI, Allahabad, Prof. Ravi S. Kulkarni (Chief Guest), Bhaskaracharya Pratishthana, Pune, (TMC past president), Prof. R. P. Tiwari (VC), CUP, Prof. R. Wusirika (Dean, In-C. (Acad.)), CUP, Dr. Sachin Kumar, CUP.

The Summer School lectures were distributed in three modules. In the first two weeks, the lectures were based on: Curves and Surfaces (Module 1) and Smooth manifolds (Module 2). In this basic course of two weeks, 24 lectures of one and half hour and 12 tutorial sessions of 1 hour were conducted. The third week was devoted to advance course entitled Surveys in Differential Geometry (Module 3). In this week, total 13 lectures of one and a half hour or one hour each were delivered by eminent Differential Geometers of top institutes of the Country. The talks are available on the YouTube Channel of CUP.

Speakers for the first two weeks: (i) Abhishek Mukherjee, Kalna College, University of Burdwan (ii) H. A. Gururaja, IISER Tirupati (iii) Bankteshwar Tiwari, DST-CIMS, Institute of Science, BHU (iv) Vikram T. Aithal, ICT Mumbai (v) Atreyee Bhattacharya, IISER Bhopal (vi) Dheeraj Kulkarni, IISER Bhopal. Tutorial sessions were handled by Soumya Dey, Postdoc., IMSc, Chennai, Sagar Kalane, Postdoc., IISER Pune. Prof. Gauree Shanker also assisted.

Speakers for the third week: (i) S. G. Dani, CEBS, Mumbai Uni. (ii) Mahuya Datta, ISI, Kolkata (iii) Ravi S Kulkarni, B. P., Pune (iv) Kapil H. Paranjape, IISER Mohali (v) Harish Seshadri, IISc. Bangalore (vi) T. Venkatesh, Math. Sci. Inst., Belagavi (vii) M. M. Tripathi, Inst. of Science, BHU, Varanasi (viii) R. K. Mishra, SLIET, Longowal, Punjab (ix) M. Prabhakar, IIT Ropar (x) Arpan Kabiraj, IIT Palakkad (xi) Rakesh Kumar, Punjabi U., Patiala (xii) Soma Maity, IISER Mohali (xiii) Saikat Mazumdar, IIT Bombay.

Valedictory Session was held on July 24, 2021 at 3:05 PM in presence of Prof. Gauree Shanker, Prof. Krishnedu Gongopadhyay, Prof. S. A. Katre (Treasurer, TMC), Prof. Ravi S. Kulkarni, Prof. Anjana Munshi (Dean Research, CUP) and Prof. R. Wusirika.

7.3 One-Day Symposium on Late Professor Shreeram Abhyankar Birthday

This Symposium was organised by Bhaskaracharya Pratishthana jointly with TMC on 22nd July 2021. The following talks were held online (available on YouTube Channel of B.P.)

1. Prof. Steven Spallone, IISER, Pune. Title: A Chinese Remainder Theorem for Partitions.
2. Prof. Steven Dale Cutkosky, University of Missouri. Title: Towards resolution of singularities in positive characteristic. (Article based on this talk is available in this issue.)
3. Prof. Avinash Sathaye, University of Kentucky. Title: Revisiting the Jacobian Problem.

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8. Problem Corner

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In the earlier issues we presented some challenging problems from Analysis, Combinatorics, Functional Equations, Graph Theory, Geometry and Number Theory. Solutions of the problems only from the Combinatorics, Geometry and Number Theory were received from the readers. In the July, 2021 issue, we posed a problem based on Graph Theory. We have received two partial solutions from Prof. Jagannath Nagorao Salunke of Latur (MH), Former Professor, School of Mathematical Sciences, SRTM University, Nanded (Maharashtra) and Meenu Mohan, Student, IISER Bhopal and a complete solution from Rosna Paul, a research scholar at Technical University, Graz, Austria which is presented below.

Also, in this issue we pose a problem from Combinatorics for our readers. Readers are invited to email their solutions to Udayan Prajapati (udayan64@yahoo.com), Coordinator, Problem Corner, before 10th December, 2021. Most innovative solution will be published in the subsequent issue of the bulletin.

Problem posed in the previous issue: A simple connected graph has 2020 vertices and each vertex has degree 3. Find the largest positive integer k satisfying: There exist k vertices, no two of them joined by an edge such that if they are deleted, the resulting subgraph is still a connected graph.

A solution by Rosna Paul: We will prove that $k = 505$.

We first prove that for every graph satisfying the given conditions, $k \leq 550$.

We have $n = 2020$ vertices in which each of them has degree 3. By a known result in Graph theory, $2\times$ the number of edges in the graph = sum of degrees of vertices = $3 \times 2020 = 6060$.

So, the number of edges in the graph = 3030.

Suppose you have deleted k vertices, no two of them joined by an edge, such that the remaining graph is connected. Let G_0 be the graph obtained after deleting the k vertices as mentioned, then as a consequence of criteria for connectedness, the number of edges in $G_0 \geq n - k - 1$.

Since none of the vertices in the collection share edges, the number of edges deleted from the original graph will be 3k. Hence the number of edges in $G_0 = 3030 - 3k \geq 2020 - k - 1$.

Thus, $1010 \geq 2k - 1$. So, $2k \leq 1011$. The largest positive integer k satisfying this inequality is 505.

Hence, $k \leq 505$ ....(1)

We now give an example of a graph for which $k = 550$ is attained.

Construct a graph on 2020 vertices as shown in Figure -1. The vertices belonging to the collection of k vertices are denoted by red color. These red vertices get deleted along with the red edges and you can see that the remaining black edges make the graph connected. Since one vertex from each group of 4 vertices are deleted, the value of k is $2020/4 = 505$. It is also clear that no other vertices could be deleted since all of them have exactly 1 red edge and hence deleting any additional vertex will violets the condition imposed on the set of deleted points. Thus we have proved that $k \geq 505$ ....(2)

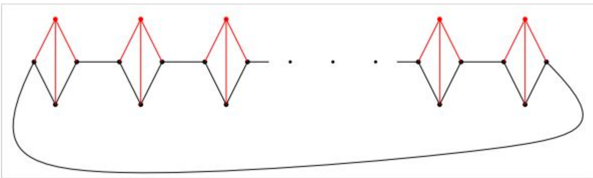

Figure-1: Total 505 rhombuses

Combining the inequalities (1) and (2), we get $k = 505$.

We can easily generalize this for a simple connected graph having $n = 4m$ vertices for any positive integer m and prove that in this case maximum number of vertices that can be deleted adhering to the condition mentioned above is m. A proof is left to the readers.

Problem for this issue

If a rectangle can be divided into 20×3 unit squares, find the number of ways of tiling the rectangle using rectangles of size 2×1 or 1×2.

□□□

9. International Calendar of Mathematics events

Ramesh Kasilingam
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November 2021

- November 3-5, 7th International Conference on Algorithmic Decision Theory (ADT 2021), Toulouse, Capitole, France. *https://www.irit.fr/ADT2021/*
- November 14, 2021, "20th Triangle Lectures in Combinatorics", University of North Carolina, Greensboro, NC, (ONLINE, hosted on Zoom). *https://wp.math.ncsu.edu › tlc*
- November 14-16, 2021, International Workshop on Domination in Graphs (IWDG-2021), Indian Institute of Technology, Ropar, India. *https://faculty.math.illinois.edu › west › meetlist*
- November 15-19, 2021, Conference: L-functions and Iwasawa theory, Station biologique de Roscoff, France. *http://www.numbertheory.org › ntw*
- November 20, 2021, 38th Annual Cascadia Combinatorial Feast, University of Victoria, Victoria, BC, Canada, (formerly Combinatorial Potlatch, Virtual on Zoom). *https://faculty.math.illinois.edu › west › meetlist*

December 2021

- December 1, 2021 - January 10, 2022, Winter Conference of Dynamicists, Winter Park Resort (Colorado, US). *utdallas.edu/ makarenkov/winter-dynamicists/*
- December 1-3, 2021, International Conference on Mathematics, Online Conference, An Istanbul meeting for world Mathematicians. *icomath.com/index.php*
- December 11-12, 2021. Interactions in Complex Geometry Workshop Vanderbilt University, Nashville, TN. *my.vanderbilt.edu/icg2021/*
- December 13-17, 2021, 43rd Australasian Combinatorics Conference (43ACC) (postponed from Dec 13-17, 2020). *https://43acc.ms.unimelb.edu.au*
- December 17-19, 2021, 10th International Conference TIME2021 (Technology & Innovations in Math Education) For Math Teachers and Educators at School and College level, Jointly hosted by: math4all & Unison World School. *http://sites.google.com/view/timeconference2021*
- December 21- 23, 2021, 15th International Conference of IMBIC on "Mathematical Sciences for Advancement of Science and Technology" (MSAST 2021). Online mode (platform Google Meet). *https://globaljobslive.com/15th-international-conference-msast-2021-december-21-23-2021/*

January 2022

- January 9, 2022 - January 12, 2022, ACM-SIAM Symposium on Discrete Algorithms (SODA 22), Westin Alexandria Old Town Alexandria, Virginia, U. S. *www.siam.org/conferences/cm /conference/soda22*
- January 10-14, 2022 Holistic Design of Time-Dependent PDE Discretizations Providence, United States. *https://icerm.brown.edu/topical_workshops/tw-22-hdtd/*
- January 20-21, 2022, Connections Workshop: The Analysis and Geometry of Random Spaces, Mathematical Sciences Research Institute, Berkeley, California. *www.msri.org/workshops/968*
- January 27-28, 2022 , 5th International Conference On Current Scenario In Pure And Applied Mathematics, Kongunadu Arts and Science College (Autonomous) Coimbatore - 641 29 Tamil Nadu, India. *sites.google.com/kongunaducollege.ac.in/iccspam2022 /home*

□□□

10. IMU related newsletters

Courtesy: T. R. Ramdas
Professor Emeritus, Chennai Mathematical Institute, Kelambakkam 603103
Email: ramdas@cmi.ac.in

The International Mathematical Union (IMU) and some of its Commissions and Committees issue following newsletters regularly throughout the year.

10.1 IMU News

IMU News is the rebranded newsletter of the IMU (known previously as IMU-Net). The first issue of the newsletter under the name IMU News appeared in January 2021. Previous iterations of the newsletter remain accessible via the *IMU News archive*.

The newsletter aims to improve communication between the IMU and the worldwide mathematical community, by reporting decisions and recommendations of the IMU, and highlighting issues that are under discussion.

In addition, IMU News will report on major international mathematical events and developments, and on other topics of general mathematical interest. Feedback from readers is welcome. One may read the latest issues at: *IMU News 109: September 2021*.

10.2 ICMI Newsletter

ICMI News is the Electronic Newsletter of the International Commission on Mathematical Instruction (ICMI), started in 2007. This newsletter aims at improving communication between ICMI and the worldwide community interested in mathematics education, informing about actions and recommendations of ICMI, highlighting issues that are under discussion and reporting about ongoing activities.

In addition, ICMI News reports on major activities by the ICMI Affiliated Study Groups (HPM, PME, IOWME, WFNMC and ICTMA), on major international events related to mathematics education and on other topics of general interest to the community of educational researchers, curriculum designers, educational policy makers, teachers of mathematics, mathematicians, mathematic educators and others interested in mathematical education around the world. Feedback from readers is welcome.

One may click here: *ICMI NEWS Archive* for all issues of the ICMI newsletter.

10.3 CWM Newsletter

CWM newsletter is the Electronic Newsletter of the Committee for Women in Mathematics. It reports Interview of CWM members, News from CWM, Other News and Announcements as the awards of important prizes to women mathematicians, the celebration of the International Day of Women and Girls worldwide, and information about the various activities undertaken by CWM.

CWM Newsletter is published twice a year, starting from May 2019. Latest issue published was Issue 5, May 2021. Suggestions for CWM News items can be sent to cwm.info@mathunion.org.

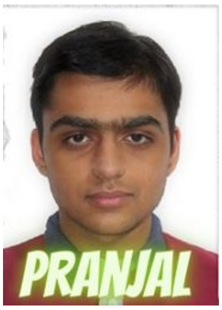
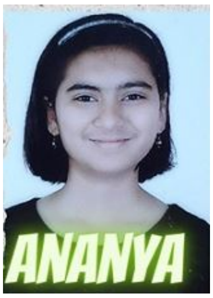
Source: A circular (email) issued by IMU secretary dated 23 September, 2021.

□□□

The 62nd International Mathematical Olympiad (IMO),

Hosted by Russia during July 14-21, 2021

Team India

	**Pranjal Srivastava** **from Bengaluru** Gold Medal – IMO, 2021 Gold Medal – IMO, 2019 Silver Medal – IMO, 2018 First Indian student to win two gold medals in IMO.		**Anish Yogesh Kulkarni** **from Pune** Silver Medal – IMO, 2021 Missed the Gold medal by just one point. Missed the IMO Team, 2019 by only few marks.
	**Ananya Rajas Ranade** **from Pune** Bronze medal – IMO, 2021 Silver medal – EGMO, 2021		**Suchir Kaustav** **from Ghaziabad** Bronze medal – IMO, 2021
	**Rohan Goyal** **from New Delhi** Bronze medal – IMO, 2021 Bronze medal – APMO-2020		**Vedant Saini** **from Chandigarh** Youngest in the Indian team with further prospects open at IMO.

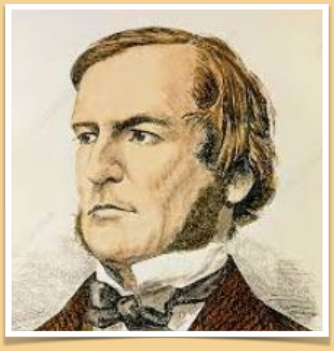
Result India – IMO, 2021

Year	Contestant [♀♂][←]	P1	P2	P3	P4	P5	P6	Total	Rank		Award
									Abs.	Rel.	
2021	Rohan Goyal	6	0	1	0	6	0	13	268	56.80%	Bronze medal
2021	Suchir Kaustav	7	1	0	0	0	7	15	180	71.04%	Bronze medal
2021	Anish Yogesh Kulkarni	5	1	0	7	7	3	23	53	91.59%	Silver medal
2021	Ananya Rajas Ranade	2	0	0	7	7	0	16	167	73.14%	Bronze medal
2021	Vedant Saini	2	0	0	6	0	0	8	348	43.85%	
2021	Pranjal Srivastava	7	0	3	7	7	7	31	17	97.41%	Gold medal

Congratulations and best wishes for a bright career to all the members of the team.

Évariste Galois (25 Oct. 1811 – 31-May 1832)

A French mathematician and Political activist.
Determined Necessary and sufficient conditions for a polynomial to be solvable by radicals. First to use the word Group in Algebra. Developed concept of a normal subgroup and a finite field. Laid a foundation for Galois theory. Contributed to Abelian integers and continued fractions.



George Boole (2 Nov. 1815 - 8 Dec. 1864)

A English mathematician, Philosopher and logician.
Contributed to the theory of analytical transformations, invariant theory, and the theory of linear differential equations. Regarded as an inventor of Boolean algebra which formed the basis of all modern electronic digital computers.

John von Neumann (28 Dec. 1903 - 8 Feb 1957)

Hungarian-American mathematician, engineer, computer scientist, Nuclear physicist, economist, chemist. One of the conceptual inventors of stored program digital computer. Developed MANIAC, the then fastest computer. Contributed to the field of mathematical logic, the foundation of quantum mechanics, economics and game theory.

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