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A Special issue in Memory of
Padma Bhushan Prof. R. P. Bambah

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Editors-in-Charge

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The Mathematics Consortium

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Presenting a special issue in memory of Prof. R. P. Bambah

Professor R. P. Bambah who passed away in May last year, missing being a centenarian by just a few months, has been one of the key figures in the shaping of mathematics in the early decades of Independent India. Apart from his own mathematical contributions in the area of Geometry of Numbers at a commendable level, he has left a lasting legacy embodied in what has come to be termed as the Punjab School of mathematics, as also on the wider platform of Indian mathematics, both through academic and administrative leadership roles that he played with remarkable acumen and efficacy. Tributes were offered to him in an obituary note, in the July 2025 issue of the TMC Bulletin, and a mention was also made of a plan to bring out a special issue of the Bulletin in April 2026 in his honour; here then is the culmination of the idea mooted at the time.

We, the editors appointed to take charge of the issue, are glad and thankful for being entrusted with the task. It has been our endeavour in the course of working on it, to bring out the essential features of the contribution of Prof. Bambah in various respects, for the benefit of the readership of TMCB and the wider mathematical community. The issue contains articles highlighting his mathematical work, the mathematical ideas he promoted by building up a school on Geometry of Numbers, his role as an educator and administrator, and above all the human angle of his personal interactions which have been instrumental in promoting mathematics.

The first two articles are devoted primarily to an exposition of his mathematical contributions. Article 1 by Hans Gill, who was one of his first students, begins with an overall introduction to Prof. Bambah and the School, and gives an exposition on some of his early work on the Ramanujan τ function, the twin issues of packings and coverings in the theory of Geometry of numbers and related topics. In Article 2, one of us (MR) who had the privilege of being a second generation member of the School, presents an account of the work on Minkowski's Conjecture on the products of nonhomogeneous linear forms, Watson's conjecture on indefinite quadratic forms, and related areas, carried out by him and the School inspired and led by him.

In Article 3, Martin Henk and Jörg M. Wills pay tribute to Prof. Bambah, with an update on a work they did following up a theme introduced by Prof. Bambah.

Article 4 is by Bindu Bambah, a physicist daughter of Prof. Bambah, who in his final years reviewed along with him the developments around the ideas that he had worked on over his career. The article recounts the perspective on them, that she shared with Prof. Bambah.

In Article 5, Sudesh Kaur Khanduja, a younger colleague from the Centre for Advanced Study in Mathematics that he nurtured at the Panjab University, reminisces on her interaction with him as her mentor, providing insights into his interactions with colleagues.

Apart from his contributions in academic pursuits, one of the highlights of Prof. Bambah is the role played by him as an administrator, having served, in particular, as Vice Chancellor of Panjab University for many years. In Article 6, Arun Grover, who had the privilege of serving in the same position about two decades later, as well as the benefit of interaction with him over extended periods in various contexts reminisces over his experiences.

In Article 7, M.S. Raghunathan gives a brief account of his interaction with Prof. Bambah, highlighting the latter's role in shaping the mathematics in early years of independent India, calling him Bhishma Pitamaha of modern Indian mathematics.

Article 8 by Surinder Pal Singh Kainth presents interesting psychological reflections on the life of Prof. Bambah and in Article 9 one of us (SGD) provides glimpses of Prof. Bambah's viewpoints as a mathematician and educator, as evidenced from his addresses in various contexts.

It is a pleasure to present the issue to the readers. We thank all the authors for their valuable contributions. We also thank Professors Vijay Pathak and S. A. Katre for their help in editing the issue, the designers Mrs. Prajkta Holkar and Dr. R. D. Holkar, and all those who have directly or indirectly helped us in the timely production of this special issue dedicated to Professor Bambah.

Editors-in-charge

1. On the Life and Mathematical Contributions of Professor R. P. Bambah - I

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Professor Ram Prakash Bambah, who passed away recently, just a few months short of completing 100 years, has left behind an enviable legacy. He was an eminent mathematician, enthusiastic teacher, great researcher, an educationist of renown, an able administrator and, above all, an admirable human being. The aim of this article, together with Part II by Madhu Raka, is to give an overview of his personality and his mathematical contributions.

1.1 Introduction

R. P. Bambah was born on September 30, 1925 at Jammu. He had his school education in Wazirabad and Sialkot, Punjab, now in Pakistan. After passing his matriculation examination in 1939 with a top rank, he joined Government College, Lahore. He stood first in the University in FA, BA, BA (Hons) and in MA (Mathematics). In MA, he set an unbeatable record by scoring 600 marks out of 600. Professor Bambah appeared for his M.A. examination in 1945. On January 1, 1946, while awaiting for the declaration of his result, he visited Government college, Hoshiarpur for an interview for the post of lecturer on leave vacancy, after an informal invitation from the college. He was told to join immediately and start teaching. As he had not come prepared for staying more than a few hours at Hoshiarpur, he showed his hesitation due to obvious difficulties. Professor Hansraj Gupta, who was the head of the mathematics department, invited young Bambah to stay with him as a family member and put all facilities he had, at his disposal. This was the beginning of their long lasting friendship and a very warm relationship, which was the envy of many and which contributed substantially to the progress of the Mathematics department at Panjab University, Chandigarh in its initial years of formation.

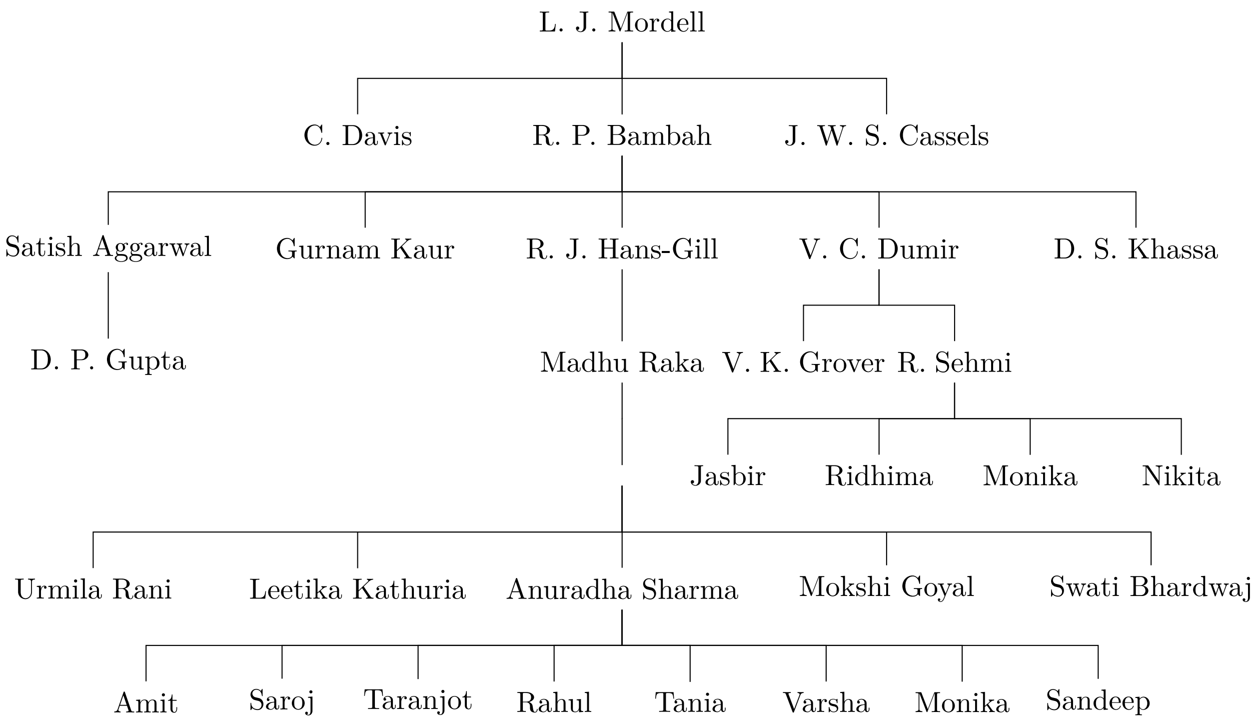
On 1st April 1946, soon after the declaration of his M.A. result, he joined as a research student at Government college, Lahore with a University scholarship of Rs. 80 pm. He was fortunate to have the eminent Number Theorist Professor S. Chowla as his mentor. In a very short period of less than a year he produced surprising number of papers and significant research work. Some of the problems that he touched upon, the questions he asked still await answers. As is well known, it is very important to ask significant questions in any scientific field. It is more so in Mathematics. The famous Mathematician Georg Cantor (1845-1918) wrote his Ph.D. thesis on the topic **"In Mathematics the art of asking questions is more valuable than solving problems"**. As I shall make this humble attempt to describe the mathematical contributions of Professor Bambah, it will be clear that he had always possessed, in abundance, both the art of asking questions and the ability to solve problems. Of course it is not possible to know and to describe his numerous invaluable contributions to many discussions and seminars, formal or informal, in which he freely asked questions, gave his guidance, suggestions and advice. His participation in a seminar or a conference always encouraged the participants directly or indirectly and made the event more significant.

Due to the political situation in 1947 he moved to Delhi and joined the Physics department of Delhi University as a lecturer in April, 1947. Next year he was elected as an 1851 Exhibition scholar from India for undertaking research in England. This was a very prestigious scholarship which was awarded by the British government to only one science scholar from India each year. He proceeded to Cambridge in August 1948 and worked under the guidance of the eminent mathematician L. J. Mordell at Saint Johns College, University of Cambridge. He got his Ph.D. in 1950. The title of his Dissertation was "Some results in the Geometry of Numbers".

The scholarship was for three years. Since he completed his Ph.D. in two years, he availed of third year of the scholarship as post doctoral fellow at University College, London. In recognition of his work, he was elected Fellow of Saint John's college, Cambridge in 1952. He was the 6th fellow from India to be elected for it. Same year he was also invited to the Institute for Advanced Study, Princeton as a temporary member. He was elected Fellow of INSA in 1954. He has published more than 70 papers in research journals and has edited a book on number theory published by INSA. He had 5 Ph.D. students and 26 descendants. He collaborated with 15 mathematicians, including many celebrated names in the area.

Professor Bambah has been decorated with many awards and honours. He was President of the Indian Mathematical Society during 1969. He was elected Fellow of INSA in 1955, Fellow of Indian Academy of Sciences in 1974, Fellow of National Academy of Sciences in1978 and Fellow of TWAS (now known as The World Academy of Sciences) in 1993. He was the vice president of INSA during 1979-80; and UGC National fellow for two years (1973-75). He was President of the Mathematical Section of ISCA during 1973 and its General President during 1983-84. He was awarded the Ramanujan Medal of INSA (1979), Distinguished Service Award by the Mathematical Association of India (1984), Meghnad Saha Award by UGC (1986), Padma Bhushan (1988), Ramanujan Memorial Lecture Award of the Indian Mathematical Society (1993), Ramanujan Birth Centenary Award of ISCA (1994), Jawaharlal Nehru Birth Centenary Lecture Award of ISCA (1997) and Aryabhata Medal of INSA (1998). In 2014, he was conferred with the degree of Doctor of Science (*honoris causa*) by Panjab University. He has been on the editorial boards of the Journal of Number Theory, Journal of Indian Mathematical Society, Mathematics Student and Indian Journal of Pure and Applied Mathematics.

Genealogy of Prof. Bambah



1.2 Prof. Bambah as my mentor

In 1961, optional one-year courses in Number Theory and Modern Algebra were introduced in Panjab University and its affiliated colleges, thanks to the efforts of Professor Bambah and Prof. Hansraj Gupta. There was a lot of resistance from colleges to the introduction of these new courses. Newspapers mentioned that ‘Bambah Mathematics’ had been introduced which the college teachers found hard to understand and teach. I was lucky that in Government College Ludhiana, where I was studying for M.A., both options were introduced. While trying problems in Algebra there were some which I could not solve and my teacher could not help in this task. Prof. K. R. Chaudhary, who was head of the department, suggested that I should take help of Prof. Bambah. He got me an appointment with him on a Saturday afternoon in Feb 1962. I arrived promptly from Ludhiana but Prof. Bambah was busy with some university meeting whole afternoon. Later in the evening he, alongwith Mrs. Bambah, visited me at the house of my relatives with whom I was staying. He expressed regrets for keeping me waiting and suggested that I could come to his house in the evening for discussions. He patiently resolved my difficulties till late in the evening. A couple of these could not be solved at that time and he promised to discuss these later on my next visit or by correspondence. I was very impressed by his method of approaching the problems, hard working nature and commitment to his work.

After my M.A. examinations, I applied for a post of assistant lecturer at Kurukshetra University. I appeared for an interview, held at Hans Raj College, Delhi. I was a little shocked to see Prof. Bambah on the Board, since he had reputation for being very strict and setting high standards. He immediately started asking me why I wanted to join a teaching post and not go for research. I replied that I had not seen any advertisement for research scholarships and that I am thinking of going abroad for research after teaching for a couple of years. (I had heard about Professor I. B. S. Passi doing that, and it had seemed to me a good plan for my career.) Professor Bambah advised not to waste time in a teaching position and to take up research at the earliest. As I did not respond positively, he was very disappointed, and the interview ended there. Eventually I got a letter of appointment at Kurukhsetra University, but in the meantime I got messages from Prof. Bambah to join research in his department. He got the impression that I wanted to join a teaching job because of financial needs and he conveyed to me through Prof. O. P. Baghai that he would try to arrange scholarship for me as soon as possible. Eventually, in August 1962 I joined Mathematics Department as a Research Scholar.

At Chandigarh, the research activities were very well organized. Gurnam Kaur and Satish Kumar Aggarwal were my seniors, Vishwa Chander Dumir and Harsh Anand joined along with me. There were several seminars : Geometry of Numbers, General Topology, Algebraic Number Theory, Diophantine approximation. All of us and most of the teachers also attended these seminars. By the summer of 1964, students working with Prof. Bambah had almost finished their Ph.D. work. At that time Prof. Bambah decided to visit Ohio State University and three of his students (Satish Aggarwal, V C Dumir and myself) accompanied him and obtained their Ph.D. degrees from Ohio State University.

After returning to Panjab University in 1968, Prof. Bambah spent all his energy in building the Math Department which had been recognized as Advanced Centre of Mathematics by UGC. He was successful in inducting several good faculty members and took keen interest in mathematical activities organizing seminars, conferences and discussions.

1.3 Prof. Bambah's Mathematical work

Most of the mathematical work of Prof. Bambah was published during 1946 to 2000. In this part, I will attempt to describe some of his early work on Ramanujan function $\tau(n)$, packings and coverings, Minkowski's second theorem and minimal density of maximal packing. In Part II, Madhu Raka will present a summary of his work on Minkowski's Conjecture on the product of n nonhomogeneous linear forms, Watson's conjecture on indefinite quadratic forms and some related areas.

Prof. Bambah's first paper [1] (1946) is on complete primitive residue sets. Determination of complete primitive residues is a basic problem in Number Theory. Chowla and Vijayaraghavan [35] had just proved that if $n = 2$ or if n has no primitive roots then there exist suitable complete primitive residue sets $r_1, r_2, ..., r_h$ and $S_1, S_2, ..., S_h$ such that $r_1S_1, r_2S_2, ..., r_hS_h$ is also a complete primitive residue set. Bambah found 2^{2m-3} such distinct systems for $n = 2^m$, $m \geqslant 3$. (Incidentally, the Chowla and Vijayaraghavan paper appeared in 1948.)

The legendary Indian mathematician **Srinivasa Ramanujan (1887-1920)** introduced the function $\tau(n)$ as the coefficient of x^n in the series expansion of

$$x[(1-x)(1-x^2)\cdots(1-x^n)\cdots]^{24}.$$

Ramanujan worked extensively with this function. He made tables of values of $\tau(n)$, found many properties and made several conjectures.

G. H. Hardy gave a series of lectures on Ramanujan's work during 1936-1940. The tenth lecture was devoted entirely to this function. In the introduction of this lecture Hardy says

> I shall devote this lecture to a more intensive study of some of the properties of $\tau(n)$, which are very remarkable and still very imperfectly understood. We may seem to be straying into one of the backwaters of mathematics, but the genesis of $\tau(n)$ as a coefficient in so fundamental a function compels us to treat it with respect.

R. P. Bambah writes in the biographical memoir [33] on S. Chowla (1907-1995) (published by INSA)

> Because of the later work of many powerful mathematicians, $\tau(n)$ occupies a very central place in main stream mathematics because of its connection with modular forms, elliptic curves and so on, which play an important role in modern mathematics and in fact, have been crucial in the solution of Fermat's last theorem.

In his second paper Bambah [2] (1946) proved two results:
Theorem A. If n is odd, then $\tau(n) \equiv \sigma_3(n) \pmod{32}$.
Theorem B. For all n, $\tau(n) = n\ \sigma_9(n) \pmod{25}$. Here $\sigma_k(n) \equiv \sum_{d|n} d^k$.

Bambah and Chowla proved several congruences for $\tau(n)$ modulo 2, 3, 5, 7, and 691. About 12 papers followed, on $\tau(n)$, in quick succession: [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]. Two of these papers are also in collaboration with H. Gupta and one with D. B. Lahiri. Some sample congruences are

- If $5 \nmid n$ then $\tau(n) \equiv 5n^2\ \sigma_7(n) - 4n\ \sigma_9(n) \pmod{5^3}$.
- If $3 \nmid n$ then $\tau(n) \equiv (n^2 + k)\ \sigma_7(n) \pmod{3^4}$, with $k = 0$ or 9 according as $n \equiv 1$ or $2 \pmod 3$.
- $\tau(n) \equiv \sigma_{11}(n) \pmod{2^8}$ when n is odd.
- $\tau(n) \equiv 0 \pmod{2^5.3^2.5^2.7.23.691}$ for almost all n.

In 1947, Chowla [36] proved that for any positive integers a, b, c, d, e, f,

$$\tau(n) \equiv 0 (\text{mod } 2^a 3^b 5^c 7^d 23^e 691^f)$$

for almost all n. The result is of special significance in the theory especially in the context of a renowned result due to P. Swinnerton-Dyer, P. Deligne and J. P. Serre that the congruences like these can be nontrivial only modulo powers of 2, 3, 5, 7, 23 and 691.

It is a well-known theorem, due to Lagrange, that every positive integer is a sum of four squares. There is also a good understanding of integers which can be expressed as a sum of three squares. What about those which are sums of two squares? How are they distributed? One of the early papers of Bambah and Chowla [14] (1947) concerns the above question, which is readily communicable. They used a straightforward construction to prove the existence of a constant C such that for $x \geqslant 1$ there is at least one integer between x and $x + Cx^{1/4}$, which can be expressed as a sum of two squares. They remarked that this is very far from the probable truth. However, no essential improvement on this result has been obtained so far though it is over 75 years since the result. In the meantime several authors [S. Uchiyama [70](1965), P. H. Diananda [41],[42],[43](1966, 67, 68) and L. J. Mordell [59](1969)] have worked on the value of C and/or generalization of this result. Hooley [52](1971), I. Richards [60](1982), E. Shiu [67](2013) and Jameson [53] (2019) have proved some related results.

Prof. Bambah's Ph.D. work

The term Geometry of Numbers was coined by the German mathematician Hermann Minkowski (1864-1909). Although various number-theoretic problems had been studied by Lagrange, Gauss and other mathematicians and geometrical methods had been used to solve arithmetical problems, it was Minkowski who observed that using geometrical methods can lead to very simple proofs of the existence of solutions in integers (not all zeros) of some inequalities. One has only to verify if the n-dimensional body defined by the inequality has certain properties. Soon these methods developed into a major branch of Number Theory. Many mathematicians tried their hand on the exciting problems. Bambah was at the right place, among the right people, in the "golden age" of Geometry of Numbers.

Let $A_1, \cdots, A_n$ be n linearly independent points in the Euclidean space $\mathbb{R}^n$. The set $\Lambda = \{u_1 A_1 + \cdots + u_n A_n : u_i \in \mathbb{Z}\}$ is called a **lattice** in $\mathbb{R}^n$ and $A_1, \cdots, A_n$ is called a **basis** of the lattice. Let $\mathbf{A}$ be the matrix having $A_1, \cdots, A_n$ as its columns. The **determinant of** Λ, denoted by $d(\Lambda)$, is defined as $|\det \mathbf{A}|$. It is easy to see that $d(\Lambda)$ is independent of the choice of basis of Λ. A lattice Λ is said to S-**admissible** for a set S in $\mathbb{R}^n$ if it has no point other than O in the interior of S. The **critical determinant** $\Delta(\mathcal{S})$ of a set $\mathcal{S} \subset \mathbb{R}^n$ is defined to be the infimum of $d(\Lambda)$'s, over all $\mathcal{S}$-admissible lattices Λ, whenever there exist any, and to be ∞ otherwise.

A fundamental problem in Geometry of Numbers is to determine the critical determinant of a given set. For convex sets centered at O in two dimensions, general results are known, which at least theoretically give the critical determinant. It is known that critical determinant of a convex domain $\mathcal{S}$ with center O in $\mathbb{R}^2$ is equal to one fourth of its area if and only if $\mathcal{S}$ is either a parallelogram or a symmetrical hexagon with center O. But for non-convex regions the problem poses substantial difficulties. Techniques have to be found and details worked out for different regions. Mordell described a method of determining critical determinant of a star body $\mathcal{S}$ which is not necessarily convex. The method involving guessing the value of the critical determinant of $\mathcal{S}$ together with critical lattices of $\mathcal{S}$. Suppose the guess is Δ, Λ respectively. Then it is enough to prove that every lattice of determinant Δ has a point $\neq O$ in $\mathcal{S}$ or on the boundary of $\mathcal{S}$. Using this method, critical determinant of various bodies have been found. The details depend upon each body under consideration. The difficulty increases as we go to higher dimensions.

In his Ph.D. thesis Bambah developed a method for determining the critical determinant of non-convex star regions with hexagonal symmetry. He used this to obtain the critical determinant of several types of regions and, in particular, of the region $|f(x,y)| \leqslant 1$, where $f(x,y)$ is a binary cubic form of positive discriminant, which was originally obtained by Mordell. He also obtained results for special non-homogeneous binary quadratic forms and for non-homogeneous cubic forms. Four papers were published from his thesis in 1951. One appeared in *Philosophical Transactions Royal Society of London* [15], two in *Acta Mathematica* [16], [17] and one in *Proc. Cambridge Phil. Soc.* [18].

Prof. Bambah's work on Lattice Covering problems

Historically, the theory of packings and admissibility had attracted much more attention than the analogous concept of coverings. Bambah developed the theory of coverings and answered several general questions about coverings, which are analogues of the corresponding results on critical determinants and packings. To give an exposition of this we first introduce some definitions. Let S, Σ be subsets of $\mathbb{R}^n$ and let $(S, \Sigma) = \{S + A : A \in \Sigma\}$ denote the family of translates of S. (S, Σ) is said to be a **packing** if $S + A$ and $S + B$ do not overlap for $A, B \in \Sigma$, $A \neq B$. (S, Σ) is called a **covering** if every point of $\mathbb{R}^n$ lies in some translate $S + A$, $A \in \Sigma$. i.e.

$$\mathbb{R}^n \subseteq \sum_{A \in \Sigma} (S + A).$$

If Σ is a lattice Λ and $(\mathcal{S}, \Lambda)$ is a covering then Λ is called a covering lattice for S and (S, Λ) is called a lattice covering .

The **covering constant** of a set S is defined the supremum of $d(\Lambda)$'s over all covering lattices (0 if there is no covering lattice). Let $\mathcal{S}$ be a closed bounded measurable set with measure $V(\mathcal{S})$. Suppose Σ be a discrete set and $(\mathcal{S}, \Sigma)$ is a covering. For $\ell > 0$, let $\mathcal{B}_\ell = \{(x_1, \cdots, x_n) : |x_i| \leq \ell, 1 \leq i \leq n\}$ be a box. Let $N_\ell =$ number of sets $\mathcal{S} + A, A \in \Sigma$, which meet $\mathcal{B}_\ell$. Define

$$\theta(\mathcal{S}, \Sigma, \ell) = \frac{N_\ell V(\mathcal{S})}{V(\mathcal{B}_\ell)} = \frac{N_\ell V(\mathcal{S})}{2^n \ell^n}.$$

and

$$\theta(\mathcal{S}, \Sigma) = \liminf_{\ell \to \infty} \theta(\mathcal{S}, \Sigma, \ell);$$

$\theta(\mathcal{S}, \Sigma)$ is called the **density** of the covering $(\mathcal{S}, \Sigma)$. If Σ is a lattice Λ, then covering density of $(\mathcal{S}, \Lambda)$ is denoted by $\theta_L(\mathcal{S}, \Lambda)$. It can be easily seen that

$$\theta_L(\mathcal{S}, \Lambda) = \frac{V(\mathcal{S})}{d(\Lambda)}.$$

The **covering density** $\theta(\mathcal{S})$ of $\mathcal{S}$ is defined as

$$\theta(\mathcal{S}) = \inf_{(S,\Sigma) \text{ a covering}} \theta(\mathcal{S}, \Sigma).$$

and the **lattice covering density** of $\mathcal{S}$ is defined as

$$\theta_L(\mathcal{S}) = \inf_{(S,\Lambda) \text{ lattice covering}} \theta_L(\mathcal{S}, \Lambda).$$

Therefore

$$\theta_L(\mathcal{S}) = \inf_{(\mathcal{S},\Lambda) \text{ lattice covering}} \left(\frac{V(S)}{d(\Lambda)}\right) = \frac{V(\mathcal{S})}{\sup\limits_{(\mathcal{S},\Lambda) \text{ lattice covering}} d(\Lambda)} = \frac{V(\mathcal{S})}{C(\mathcal{S})}.$$

One of the important problems in geometry of numbers is to determine the lattice covering density $\theta_L(\mathcal{K}_n)$ of the n-dimensional unit sphere $\mathcal{K}_n$. For circles, the result goes back to Dirichlet [45](1842) who proved that $\theta_L(\mathcal{K}_2) = \frac{2\pi}{3\sqrt{3}}$. **Bambah [19](1954) obtained $\theta_L(\mathcal{K}_3) = \frac{5\sqrt{5}\pi}{24}$ and also gave, in [20] (1954), an estimate for $\theta_L(\mathcal{K}_4)$ and conjectured its exact value.** This conjecture was proved by B. N. Delone and S. S. Ryshkov [40](1963) and T. J. Dickson [44](1967). Ryskov and Baranowskii [64](1975) obtained exact value of $\theta_L(\mathcal{K}_5)$.

The first non-trivial bound for $\theta_L(\mathcal{K}_n)$ for general n was obtained by Bambah and Davenport (1952)[21]. They proved that

$$\theta_L(\mathcal{K}_n) \geqslant \frac{4}{3} - \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to \infty$. This was later improved upon by Coxeter, Few and Rogers [37] (1959). Bambah and Roth [22] (1952) obtained upper bounds for the best lattice covering density of convex bodies in $\mathbb{R}^n$, which are symmetrical about the co-ordinate planes. Earlier results in this direction were due to Hlawka [51](1945) and Rogers [61](1950). In 1957, Bambah [23] defined the concept of maximal covering domain as an analogue of irreducible bodies studied by Mahler [56](1947) and Rogers [62], [63] (1947,1952). In my thesis, I [50](1967) extended their results to higher dimensions and different classes of sets.

In small dimensions, the most efficient covering of n-dimensional space $\mathbb{R}^n$ by spheres is given by the so called Voronoi form of the first type

$$(n+1)\sum_{i=1}^{n} x_i^2 - \Big\{\sum_{i=1}^{n} x_i\Big\}^2,$$

i.e. by the dual lattice A_n^* to the root lattice A_n. Ryskov and Baranowskii [65](1976) have shown that this is the most efficient lattice covering for $n \le 5$. Ryskov [66](1967) showed that A_n^* is not the most efficient lattice covering for all even $n \ge 114$ and all odd $n \ge 201$ and raised the question of finding the first dimension n for which there is a better lattice. Bambah and Sloane [24](1982) constructed lattice coverings which are more efficient than A_n^* in all dimensions $n \ge 24$ by using the covering radius of 24-dimensional Leech lattice.

Prof. Bambah's work on General coverings

A natural problem in geometry of numbers is to determine whether the best lattice covering density $\theta_L(K)$ of a given body K is equal to its best general covering density $\theta(K)$. Kerschner [54](1939) proved that this is so for a circle. Fejes Tóth [69](1950) gave a proof of $\theta(K) = \theta_L(K)$, for two-dimensional symmetric convex domains. Bambah and Rogers [25](1952) gave alternative versions of Fejes Tóth's construction. Simplified proofs of this result were given by Bambah, Rogers and Zassenhaus [26](1964) and by Bambah and Woods [27](1968). They also showed that $\theta(C) = \theta_L(C) = \theta(K)$, for the cylinder C of height 1 with base K, which is a symmetric convex domain. Their methods lead to the study of finite coverings, a field on which lot of work has been done by later mathematicians.

Stein [68](1972) proved that $\theta(S) = \theta_L(S)$ does not necessarily hold for star bodies. For this, he gave an example of a non-symmetric star body in five-dimensional space and a symmetric star body in ten-dimensional space. Bambah, Dumir, and Hans-Gill [29](1977) gave examples to show that there exist star domains (symmetric as well as asymmetric) in the plane $\mathbb{R}^2$ for which $\theta(S) < \theta_L(S)$. Another interesting and basic question is whether $\theta(K) = \theta_L(K)$ for the three-dimensional sphere K. Bambah and Woods [30](1971) proved that double lattices (sets that consist of a union of a lattice and one of its translates) do not give better coverings than the lattices for the three-dimensional spheres. The general question is still open; (the analogous problem for packings is known as Kepler's conjecture and was resolved by T. C. Hales [49] using lot of computer work).

Prof. Bambah's work on Minkowski's second theorem

Let $\mathcal{K}$ be a symmetric convex body in $\mathbb{R}^n$ with center O and Λ be a lattice in $\mathbb{R}^n$. Let, for $1 \le r \le n$

$$m_r = \inf\{\lambda : \lambda\mathcal{K} \text{ contains } r \text{ linearly independant points of } \Lambda\}. \tag{1.1}$$

Then $m_1, m_2, \cdots, m_n$ are called **successive minima** of $\mathcal{K}$ with respect to Λ.
Minkowski's second inequality states that

$$m_1 m_2 \cdots m_n V(\mathcal{K}) \le 2^n d(\Lambda). \tag{1.2}$$

This is also known as **Minkowski's Second Theorem**. This is a generalization of his fundamental theorem. Minkowski [58] himself gave a proof of the inequality (1.2) in 1896. His proof is quite difficult. It was simplified by Weyl [71] (1942) and Cassels [34](1957), and a quite different proof was given by Davenport [39](1939). However, none of these proofs was completely satisfactory. Professor Mahler, during a seminar at Notre Dame University, suggested to Bambah, Woods and Zassenhaus that it would be worthwhile to re-examine these proofs with a view to possible generalizations. Each of them then gave a proof, one based on Weyl's paper [71] and two on Davenport's [39]. Bambah, Woods and Zassenhaus [28] (1965) in a paper "Three different proofs of Minkowski's Theorem" gave three proofs which are simpler and more satisfactory. They write in their paper "In Weyl's proof all considerations are made in the quotient space determined by the lattice Λ. The aim of the first proof is to show that the existence of the quotient space is needed only to deduce the so-called monotone property, thus suggesting that the theorem is true for some point sets other than lattices. Davenport's proof of (1.2) depends on certain functions constructed in the course of the argument being continuous. He states without proof that the construction can be made to ensure this. The third proof here shows how this may be done. Earlier Siegel in lectures at New York University gave without proof a method for making this construction. The second proof shows that by working with iterated integrals instead of Jordan contents Davenport's proof can be made independent of this continuity."

These proofs have been used by A. C. Woods [72] (1966) and R. B. McFeat [55](1971) to prove some generalizations. Woods gave a generalization of Minkowski's Second inequality following Bambah's version of Davenport's proof. Woods proved that inequality (1.2) remains true if the restriction that Λ be a lattice is replaced by the weaker condition

$$\text{if } X, Y \in \Lambda, \text{ then either } X - Y \text{ or } Y - X \text{ is in } \Lambda.$$

Danicic [38](1969) gave an elementary proof of Minkowski's Second Theorem. In 2002, Martin Henk [57] gave a short proof of Minkowski's second theorem based on the ideas of Minkowski's original proof.

Prof. Bambah's work on minimal density of maximal packings

L. Fejes Toth [48] (1965) had conjectured that for the circle K, there is a maximal K-packing of radius r whose lower density is the minimal lower density and every member of which has radius r. Eggleston [47](1965) had proved this. Bambah and Woods [31] (1968) extended this result to the case when K is a centrally symmetric strictly convex body. Eggleston had also proved the result for circles forming a saturated system, with the restriction that circles in the system do not overlap. Bambah and Woods extended this result for overlapping saturated system but with two restrictions. His students V. C. Dumir and Dharam Singh Khassa [46](1973) were able to remove these conditions.

Bambah has also contributed several other results, relating to integer matrices, polar reciprocal convex bodies, divided cells, transference theorems, lower bounds for minimum distance codes, convex bodies with covering property, maximal covering sets and many other topics in discrete geometry. In particular Bambah and Woods (1994)[32] solved a problem of G. Fejes Toth on the thinnest lattice arrangement of spheres, so that each straight line meets one of these spheres.

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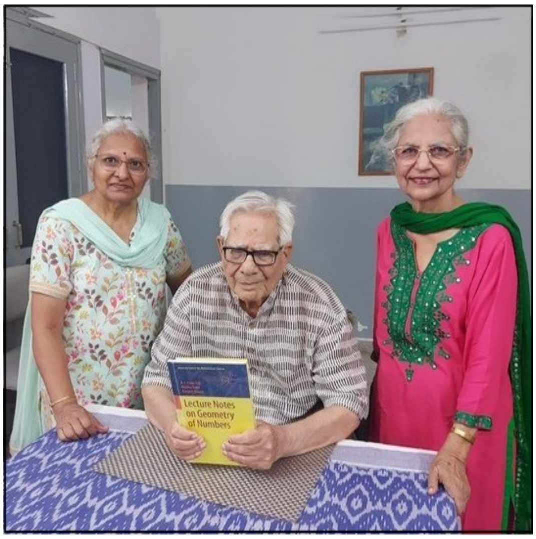

Professors R. J. Hans Gill (right) and Madhu Raka (left) presenting to Prof. Bambah a copy of their book: Lecture Notes on Geometry of Numbers, dedicated to him. (27 July 2024)

Professor Bambah, with Professors Hans Gill, S. P. S. Kainth and Dinesh Khurana, rejoicing over the award of Doctor of Science (Honoris Causa) received by Prof. Hans Gill. (20 March 2025)

2. On the Life and Mathematical Contributions of Professor R. P. Bambah - II

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In the first part of this article R. J. Hans Gill recalled some aspects of Prof. Bambah's mathematical contributions. The aim of this article is to complement it with a summary of his contributions on certain other topics (see below), after briefly going over some reminiscences, associated with him as my mentor.

2.1 Professor R. P. Bambah as my mentor

Professor Bambah was a mentor, a visionary and a statesman of Indian higher education inspiring countless students and faculty members. He was a trailblazer in Number Theory and Discrete Geometry leaving an indelible mark on the mathematical community.

My first one-to-one interaction with Professor Bambah was during my first year of Ph.D., when he gave us a course on Algebraic Number Theory. He was an excellent teacher, but one had to be very attentive in his class because of his way of writing on the black board randomly; a few seconds distraction would make us lost. After completing my course work in first year, I requested him to take me as his Ph.D. student. But I could not get that privilege because of his heavy administrative responsibilities. He advised me to consult Professor R.J. Hans-Gill (she did her Ph.D. under the guidance of Prof. Bambah and was working as a faculty in the department at that time), but he assured me of his continuous guidance and support, which he did till his end. The topic of my Ph.D. research (and hence of later work) was suggested to me by him only, when I was struggling to find one. "Why don't you start from where Vishwa (Prof. V. C. Dumir) left"- were the exact words which he said to me while climbing down the stairs of the department. Then I consulted Prof. Dumir, took his Ph.D. thesis and this is how I was drawn into Inhomogeneous Problems.

There are numerous incidents when he guided me and helped me. While writing a book " Lecture notes on Geometry of Numbers" (jointly with Prof. Hans-Gill and Prof. Ranjeet Sehmi, published by Springer in 2024) I visited him to discuss some comments, that had been received from a referee. One of the issues raised was- why we don't work with lattices on general inner product spaces rather than working only on Euclidean spaces. Prof. Bambah asked me to bring a book from second rack of his book almirah, and pointed out some relevant pages saying that there I might get answers to our queries. Such wonderful and unbelievable was his memory – he was 98 at that time and was mostly lying on the bed. During his last few months, whenever I visited him, he discussed mathematics. In fact the topic and material of my technical presidential talk " Some classical problems of Geometry of numbers" given at the Annual meeting of IMS in Dec., 2024 was suggested to me by him only. Later, on his insistence, I recited my entire talk to him, he listened attentively with his eyes closed, making comments occasionally. That night he ate to his full and slept well, as his daughter Prof. Bindu Bambah told me.

Prof. Bambah's mathematical work is published during 1946 to 2000 and involves more than 70 research papers. He did not co-author the papers from the doctoral work of his students; had he done so, as is generally the practice, the number of his publications would have been much more.

In what follows the aim of this article is to describe some of Professor Bambah's research work. In the previous article, Prof. Hans-Gill has given a summary of his work on Ramanujan function $\tau(n)$, packing and covering, Minkoski's second theorem and minimal density of maximal packings. Here I give a brief account of his later work mainly on Minkowski's Conjecture on the product of n non-homogeneous linear forms, Watson's conjecture on indefinite quadratic forms and related areas.

2.2 A conjecture of Minkowski

Hermann Minkowski [54] proved the following Theorem in 1899 (published in 1901):
Let

$$L_1(x_1, x_2) = \alpha x_1 + \beta x_2,$$
$$L_2(x_1, x_2) = \gamma x_1 + \delta x_2,$$

be two real linear forms of determinant $\Delta = \alpha\delta - \beta\gamma \neq 0$. Then given any real numbers c_1, c_2 there exist integers u_1, u_2 such that

$$|L_1(u_1, u_2) + c_1||L_2(u_1, u_2) + c_2| \leq \frac{1}{4}|\Delta|.$$

Further, equality is necessary if and only if there exist a unimodular transformation which changes the forms to $L_1 = \lambda_1 x_1,\ L_2 = \lambda_2 x_2$ for some numbers λ_1, λ_2. For these forms equality is needed if and only if $c_1 = \frac{1}{2}\lambda_1 + m_1\lambda_1,\ c_2 = \frac{1}{2}\lambda_2 + m_2\lambda_2$, where $m_1,\ m_2$ are integers. A number of proofs of this result are now available in literature. Minkowski is believed to have conjectured that the work generalizes to n variables.

Minkowski's Conjecture :

Let $L_1, L_2, \cdots, L_n$, $n \geq 2$, be n real linear forms in n variables $x_1, \ldots, x_n$ with determinant $\Delta \neq 0$. Then given any real numbers $c_1, \ldots, c_n$, there exist integers $u_1, \ldots, u_n$ such that

$$\prod_{i=1}^{n} |(L_i(u_1, \ldots, u_n) + c_i)| \leq \frac{1}{2^n}|\Delta|. \tag{2.1}$$

Further, equality is needed in (2.1) if and only if there exists an integral unimodular transformation of the variables which transforms the linear forms to $L_i = \lambda_i x_i$. For these forms equality is needed only for $c_i = \frac{1}{2}\lambda_i + m_i\lambda_i$, where m_i are integers for $1 \leq i \leq n$.

In his lecture "Recent progress in the Geometry of Numbers" at ICM-1950, H. Davenport talked in detail about the progress made to date on Minkowski's Conjecture. He summed up : "In spite of all the work I have mentioned, Minkowski's Conjecture remains unproved, and is an outstanding challenge to all who are interested in the subject."

Peter Gruber [37] (2007) in his book on Convex and Discrete Geometry states that "The conjecture is one of those seminal problems which over a century has generated numerous notions, problems and results of different types in number theory and in particular in the geometry of numbers. One such notion is that of DOTU matrices. Tools used in this context, and in particular, tools to attack the conjecture range from algebraic topology to measure and algebraic number theory."

This classical conjecture of Minkowski is so far known to be true for $n \leq 10$. The conjecture was proved by Minkowski [54] himself for $n = 2$ in 1901. Several proofs of this result have since been given (Mordell (1928, 1941, 1953), Landau (1931), Perron (1938), Pall (1943), Macbeath (1948, 1961), Sawyer (1948) and Cassels (1953)), partly in an effort to find a proof which would generalize to higher dimensions.

For $n \geq 3$, we display here the list of mathematicians who contributed to settling the conjecture.

$n = 3$	Remak [62](1923); Davenport [23](1939); Birch and Swinnerton-Dyer [14](1956); Narzullaev [55],[56](1968, 1974)
$n = 4$	Dyson [34](1948); Skubenko [65](1973); **Bambah** and Woods [9](1974)
$n = 5$	Skubenko [66](1973); **Bambah** and Woods [11](1980)
$n = 6$	C. T. McMullen [53](2005)
$n = 7$	Hans-Gill, Raka and Sehmi[42](2009)
$n = 8$	Hans-Gill, Raka and Sehmi [43](2011)
$n = 9$	Kathuria and Raka [47](2016)
$n = 10$	Kathuria and Raka [48](2022)

Geometric Interpretation of Minkowski's Conjecture:

Any lattice Λ of determinant $d(\Lambda) = |\,\Delta\,|$ in $\mathbb{R}^n$ is a covering lattice for the set

$$\mathcal{S} = \left\{ X : |x_1 x_2 \cdots x_n| \leq \frac{|\,\Delta\,|}{2^n} \right\}.$$

Further Λ is not a covering lattice for the interior of $\mathcal{S}$ if and only if Λ has a basis consisting of vectors of the type $P_i = (0, \cdots, 0, b_i, 0, \cdots, 0),\ 1 \leq i \leq n$ for $b_i \in \mathbb{R}$.

On replacing L_i by $\frac{L_i}{\Delta}$ and c_i by $\frac{c_i}{\Delta}$, we see that Minkowski's Conjecture is equivalent to the following statement:

Any lattice Λ in $\mathbb{R}^n$ with determinant 1 is a covering lattice for the set

$$\mathcal{S} = \left\{ X : |x_1 x_2 \cdots x_n| \leq \frac{1}{2^n} \right\}.$$

As mentioned earlier, several authors have tried to find different ways of solving the case $n = 2$, in the hope of finding a method which will extend to higher dimension.

For $n \geq 3$, the following approaches have been tried. Only the first has been successfully extended to prove the result for $4 \leq n \leq 10$.

I. Remak-Davenport Approach
II. Birch and Swinnerton-Dyer Approach
III. DOTU-matrix Approach by Macbeath
IV. Shapira and Weiss Approach using stable lattices.

Remak-Davenport Approach: Remak's proof (1923) for $n = 3$, which was later simplified by Davenport (1939) consists of proving the following two conjectures:

Conjecture I. For any lattice Λ in $\mathbb{R}^n$ there is an ellipsoid $\mathcal{E} = \{X : a_1 x_1^2 + \cdots + a_n x_n^2 < 1\}$ which contains no point of Λ other than O but has n linearly independent points of Λ on its boundary.

Conjecture II. If Λ is a lattice of determinant 1 and there is a sphere $\{X : |X| < R\}$ which contains no point of Λ other than O and has n linear independent points of Λ on its boundary then Λ is a covering lattice for the closed sphere of radius $\sqrt{n}/2$. Equivalently every closed sphere of radius $\sqrt{n}/2$ lying in $\mathbb{R}^n$ contains a point of Λ.

It can be easily seen that Minkowski's Conjecture (except for the determination of critical cases) follows from Conjectures I and II. Take any lattice Λ of determinant 1 and $\mathcal{E}$ any ellipsoid whose existence is implied in Conjecture I. On making a linear transformation

$$\sqrt{a_i}x_i = (a_1 a_2 \cdots a_n)^{\frac{1}{2n}} y_i,\ 1 \le i \le n,$$

which is an automorphism of the form $x_1 \cdots x_n$ and replacing Λ by the transformed lattice we can suppose that the ellipsoid $\mathcal{E}$ found in Conjecture I is a sphere, for which the hypothesis of Conjecture II holds.
The conclusion of Conjecture II implies that Λ is a covering lattice for the sphere $\mathcal{K} = \{X : |X| \le \sqrt{n/4}\}$. Now using Arithmetic-Geometric inequality

$$\mathcal{K} \subseteq \mathcal{S} = \{X : |x_1 x_2 \cdots x_n| \le 2^{-n}\},$$

it follows that Λ is a covering lattice for $\mathcal{S}$.
In 1956, Birch and Swinnerton-Dyer [14] proved the following result :

Birch and Swinnerton-Dyer Theorem :

Suppose that Minkowski's Conjecture has been proved for $n = 1, 2, \cdots, N-1$. To prove it for $n = N$ it is enough to consider only sets of forms $L_1, \cdots, L_n$ for which the homogenous minimum

$$M_H = \inf\{|L_1(X)L_2(X)\cdots L_n(X)| : X = (x_1, x_2, ..., x_n) \in \mathbb{Z}^n,\ X \ne O\}$$

is attained and is non zero.

In view of this result, we need consider only those lattices Λ for which the homogenous minimum $M_H(\Lambda) = \inf\{|x_1 \cdots x_n| : X = (x_1, \cdots, x_n) \in \Lambda, X \ne O\}$ is attained and is non-zero. So for the purpose of proving Minkowski's conjecture, one may replace Conjecture I by Conjecture I$'$.

Conjecture I$'$. For any lattice Λ in $\mathbb{R}^n$ with $M_H(\Lambda) \ne 0$, there is an ellipsoid $\mathcal{E} = \{X : a_1 x_1^2 + \cdots + a_n x_{n^2} < 1\}$ which contains no point of Λ other than O but has n linearly independent points of Λ on its boundary.

Dyson [34] (1948) gave the first proof of both the conjectures- Conjecture I and Conjecture II for $n = 4$ and hence of Minkowski's Conjecture. Dyson's proof of Conjecture I for $n = 4$ borrows powerful tools from algebraic topology. **Bambah** and Woods [9] (1974) gave an elementary proof of Dyson's theorem, without using strong tools from algebraic topology. A different proof of this result was given by Skubenko [65], [66] (1972, 1973), which also dealt with the case $n = 5$. **Bambah** and Woods found that the argument of Skubenko had gaps. Following the general strategy of Skubenko, a complete proof for $n = 5$ of Conjecture I was given by **Bambah** and Woods [11] in 1980. The proof of Conjecture II for $n = 4$ by Dyson [34] (1948) was later simplified by Cleaver [20] (1965) and Woods [70](1965a). Woods [71], [72] (1965b, 1972) gave proofs of Conjecture II for $n = 5$ and 6. Thus results of **Bambah** and Woods [11] (1980) completed the proof of Minkowski's conjecture for $n = 5$. Conjecture I, though considered more difficult in the beginning, was fully settled by McMullen [53] in 2005 for all $n \ge 3$. This completed the proof of Minkowski's conjecture for $n = 6$.

Once Conjecture I was fully settled in 2005 for all n, Professor Bambah encouraged the Chandigarh school to work on Conjecture II. Hans-Gill and Dumir started working on it by exploring Woods' method but the work was cut short because of untimely death of Prof. Dumir in 2006. Hans-Gill started a seminar again in 2007 in the department in which Ranjeet Sehmi, Sucheta and myself participated and we were successful in extending the method of Woods for higher n. Following Table displays the history of these conjectures:

	Conjecture I	Conjecture II
$n=3$	Remak[62](1923) Davenport [23](1939) McMullen [53](2005)	Remak[62](1923) Davenport [23](1939) Mahler [51](1940)
$n=4$	Dyson [34](1948) **Conjecture I′** Skubenko [65](1973) Bambah and Woods [9](1974) McMullen [53](2005)	Hofreiter [44](1933) Dyson[34](1948) Cleaver [20](1966) Woods [70](1965a)
$n=5$	Skubenko [66](1973) Bambah and Woods [11](1980) McMullen [53](2005)	Woods [71](1965b)
$n=6$	McMullen [53](2005)	Woods [72](1972)
$n=7$	McMullen [53](2005)	Hans-Gill, Raka and Sehmi[42](2009)
$n=8$	McMullen [53](2005)	Hans-Gill, Raka and Sehmi[43](2011)
$n=9$	McMullen [53](2005)	Kathuria and Raka[47](2016)
$n=10$	McMullen [53](2005)	Kathuria and Raka[48](2022)
$n \geqslant 10$	McMullen [53](2005)	

Woods used Korkine-Zolotareff [46] (1873, 1877) reduction of quadratic forms in his proof of Conjecture II for $n = 4, 5$ and 6. By this reduction, a cartesian co-ordinate system can be chosen in $\mathbb{R}^n$ in such a way that Λ has a basis of the form

$$(A_1, 0, 0, \cdots, 0), (a_{2,1}, A_2, 0, \cdots, 0), \cdots, (a_{n,1}, a_{n,2}, \cdots, a_{n,n-1}, A_n)$$

where $A_1, A_2, \cdots, A_n$ are all positive and further for each $i, 1 \le i \le n$, any two points of the lattice Λ_i in $\mathbb{R}^{n-i+1}$ with basis

$$(A_i, 0, 0, \cdots, 0), (a_{i+1,i}, A_{i+1}, 0, \cdots, 0), \cdots, (a_{n,i}, a_{n,i+1}, \cdots, a_{n,n-1}, A_n)$$

are at a distance at least A_i apart, i.e., $\min\{|P| : P \in \Lambda_i,\ P \neq O\} = A_i$.
In fact, Woods [70] (1965a) formulated the following more general conjecture from which Conjecture II follows immediately.
Conjecture III (Woods): If $A_1 A_2 \cdots A_n = 1$ and $A_i \leqslant A_1$ for each i then any closed sphere in $\mathbb{R}^n$ of radius $\sqrt{n}/2$ contains a point of Λ.
Woods [70], [71], [72] (1965a, 1965b, 1972) proved this conjecture for $n = 4, 5$ and 6. With encouragement from Prof. Bambah, Hans-Gill, Raka and Sehmi [42], [43] (2009, 2011) proved Woods' Conjecture for $n = 7$ and $n = 8$. I, with my student Leetika Kathuria [47], [48] (2016, 2022), settled Woods' conjecture for $n = 9, 10$. So courtesy McMullen and Woods, Minkowski's Conjecture is proved for $n \le 10$. In 2017, Regev et al. [63] showed that Woods' Conjecture is false for $n \ge 30$. Chen and Xu [19] (2019) proved its falsehood for $n \ge 24$. (This does not mean that Minkowski's Conjecture is false for these n). It will be of interest to find the largest value of n, $11 \le n \le 23$, for which Woods' Conjecture is true.

Birch and Swinnerton-Dyer Approach

Birch and Swinnerton-Dyer [14] (1956) used a homogenous reduction of linear forms to give a proof of Minkowski's Conjecture for $n = 3$. This proof has not been extended so far to get a proof of Minkowski Conjecture for any larger value of n. But this contains a very significant result stated earlier which has been used by various authors (Bambah and Woods, Skubenko, McMullen) to restrict their attention to Conjecture I$'$.

DOTU matrix Approach

The third approach to Minkowski's Conjecture that has been tried is via factorization of matrices. Macbeath [50] (1961) called a square non-singular matrix M a **DOTU**-matrix if it can be written as a product DOTU, where D is a diagonal matrix, O is an orthogonal matrix, U is a unimodular matrix and T is unit triangular matrix, i.e., T$= (t_{ij})$, $t_{ii} = 1$ for $1 \le i \le n$, $t_{ij} = 0$ for $i > j$. Macbeath [50] showed that every DOTU-matrix M is a Minkowski matrix, i.e., the lattice $\Lambda = MZ^n$ is a covering lattice for

$$\mathcal{S} = \{(x_1, \cdots, x_n) : |x_1 \cdots x_n| \le 2^{-n} |\det M|\}.$$

Thus Minkowski's Conjecture would follow for a given n if it can be proved that every non-singular matrix of order n is a DOTU-matrix. Macbeath remarked that it is easy to see that for $n = 2$ every non-singular matix is a DOTU-matrix. Narzullaev [55], [56] (1968, 1974) proved that this also holds for $n = 3$. Gruber [36] (1976) and Ahmedov [4] (1977) showed the existence of nonsingular matrices which are not DOTU, using tools from Algebraic Number Theory. Specific examples were given by Skubenko [67] (1981) for $n = 2880$ and Lenstra [49] (1985) for $n = 64$. Thus this approach failed to proceed further.

Shapira and Weiss Approach

Shapira and Weiss [64] (2013) showed that if all stable lattices in $\mathbb{R}^n$ have covering radius at most $\frac{\sqrt{n}}{2}$, then Minkowski's conjecture is true in dimension n. A lattice Λ of determinant 1 is called **stable** if any subgroup of Λ is of covolume at least 1. Shapira and Weiss [64] conjectured that *Covering radius of any stable lattice in* $\mathbb{R}^n$ *is at most* $\frac{\sqrt{n}}{2}$. They proved that it is true for $n \le 7$ using results on locally extremal lattices. This method has not been extended so far to get a proof of Minkowski's Conjecture for any larger value of n.

2.3 A Conjecture of Watson

Minkowski's Theorem for $n = 2$ can be reformulated in terms of non-homogeneous minima of indefinite binary quadratic forms. Given two linear forms $L_1(x, y) = \alpha x + \beta y$ and $L_2(x, y) = \gamma x + \delta y$, of determinant $\Delta = \alpha\delta - \beta\gamma \ne 0$, let $Q(x, y) = L_1(x, y)L_2(x, y) = ax^2 + bxy + cy^2$. Then $Q(x, y)$ is an indefinite binary quadratic form of determinant $D = ac - b^2/4 = \Delta^2/4$. Given any real numbers c_1, c_2 there exist reals x_0, y_0 such that $L_1(x_0, y_0) = c_1,\ L_2(x_0, y_0) = c_2$ so that

$$(L_1 + c_1)(L_2 + c_2) = Q(x + x_0, y + y_0). \tag{2.2}$$

Conversely, given an indefinite binary quadratic form $Q(x, y)$ and real numbers x_0, y_0, we can find linear forms L_1, L_2 and real numbers c_1, c_2 such that equality (2.2) holds. Minkowski's Theorem for $n = 2$ can be restated as

Theorem 1. Let $Q(x, y)$ be an indefinite binary quadratic form of determinant $D \ne 0$. Then given any real numbers x_0, y_0 there exist integers x, y such that

$$|Q(x + x_0, y + y_0)| \le \left(\frac{1}{4}|\Delta|\right)^{1/2}. \tag{2.3}$$

Equality is needed in (2.3) if and only if $Q(x, y) \sim \rho xy$, $\rho \ne 0$ and $(x_0, y_0) \equiv (\frac{1}{2}, \frac{1}{2}) (\text{mod } 1)$.

A natural question is to generalize the above result to indefinite quadratic forms in n variables. Let $Q(x_1, \cdots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$, $a_{ij} = a_{ji}$ be an indefinite quadratic form of determinant $D = \det(a_{ij}) \neq 0$. Let Q be of the type (r, s), where $1 \leq r, s < n$, $n = r + s$ and $\sigma = r - s$ being its signature. Blaney [16] (1948) showed that given any real numbers $c_1, c_2, \cdots, c_n$, there exists a constant C depending upon n and σ only such that the inequality

$$|Q(x_1 + c_1, x_2 + c_2, \cdots, x_n + c_n)| \leq (C|D|)^{1/n} \tag{2.4}$$

has a solution in integers $x_1, x_2, \cdots, x_n$. Let $C_{r,s}$ denote the infimum of all such constants C over various choices of $c_1, c_2, \cdots, c_n$. Clearly $C_{r,s} = C_{s,r}$. So we need consider non-negative signatures only. The quadratic forms Q for which equality is needed in (2.4) with $C = C_{r,s}$ for some $c_1, c_2, \cdots, c_n$ are called critical forms. Minkowski's Theorem for $n = 2$ states that $C_{1,1} = \frac{1}{4}$. For $n = 3$, Davenport [24] (1948) proved that $C_{2,1} = C_{1,2} = \frac{27}{100}$. For all n and $\sigma = 0$, Birch [15] (1958) proved that $C_{r,r} = \frac{1}{4}$ for all $r \geq 1$.
In 1962, Watson [69] determined the values of $C_{r,s}$ for all $n \geq 21$ and for all signatures σ. He proved that

$$C_{r,s} = \begin{cases} \frac{1}{4} & \text{if} \quad \sigma \equiv 0 \text{ or } 1(\bmod 8) \\ \frac{1}{3} & \text{if} \quad \sigma \equiv 2(\bmod 8) \\ \frac{1}{2} & \text{if} \quad \sigma \equiv 3(\bmod 8) \\ 1 & \text{if} \quad \sigma \equiv 4(\bmod 8). \end{cases} \tag{2.5}$$

The problem for small values of n was considered difficult. Watson[69] (1962) conjectured that (2.5) holds for all $n \geq 4$. Under Prof. Bambah's guidance and encouragement, the Chandigarh school founded by him succeeded in evaluating $C_{r,s}$ for all values of n. Dumir [26] (1967) proved Watson's conjecture for $n = 4$. For $n = 5$, the conjecture was proved by Hans-Gill and Raka [38], [40] (1979, 1980). It may be remarked that the conjectured values of $C_{r,s}$ for $n \geq 4$ depend only on the class of $\sigma = r - s(\bmod 8)$.

Following the method of Birch, Madhu Raka [58], [59], [60] (1981, 1983a,1983b) proved Watson's conjecture for all $\sigma = \pm 1, \pm 2, \pm 3, \pm 4$. Thus Watson's conjecture was settled for signatures in a complete set of residues mod 8. The final blow was given by Dumir, Hans-Gill and Woods [33] in 1994.
In 1960, Watson 68 had proved the following results :

Theorem 2. Let $f(x_1, x_2, \cdots, x_n)$ be an indefinite non-singular quadratic form of determinant $D \neq 0$ in $n \geq 3$ variables. Let $\alpha, c_1, c_2, \cdots, c_n$ be any real numbers. Suppose that one of the following conditions hold:

(i) f is incommensurable and takes arbitrary small non-zero values for integers $x_1, x_2, \cdots, x_n$

(ii) f is a multiple of a rational form that represents zero non-trivially and $c_1, c_2, \cdots, c_n$ are not all rational.

Then for any $\epsilon > 0$, the inequality

$$|f(x_1 + c_1, x_2 + c_2, \cdots, x_n + c_n) - \alpha| < \epsilon$$

has a solution in integers $x_1, x_2, \cdots, x_n$.

Meyer's theorem states that an indefinite integral quadratic form $f(x_1, x_2, \cdots, x_n)$ in $n \geq 5$ variables represents zero nontrivially, i.e., there exist integers $u_1, u_2, \cdots, u_n$ not all zero such that

$f(u_1, u_2, \cdots, u_n) = 0$. Oppenheim's conjecture can be viewed as an analogue of this statement for forms that are not multiples of a rational form. These are called incommensurable forms.

Oppenheim Conjecture [57] (1929) Let $f(x_1, x_2, \cdots, x_n)$ be an incommensurable quadratic form of determinant $D \neq 0$ in $n \geq 3$ variables. Then for any $\epsilon > 0$, there exist integers $u_1, u_2, \cdots, u_n$ not all zero such that

$$0 < |f(u_1, u_2, \cdots, u_n)| < \epsilon.$$

In 1987, Fields medalist Margulis [52] proved this famous conjecture. For a detailed study on Oppenheim's Conjecture, see Dani [22] (2008).

Using the results of Watson [68](1960) and of Margulis [52] (1987), one can restrict to primitive integral forms $Q(X)$ only, and to $(c_1, c_2, \cdots, c_n) \in \mathbb{Q}^n$ in order to prove that $C_{r,s}$ depends only on signature mod 8.

In 1994, Dumir, Hans-Gill and Woods [33] proved that $C_{r,s} = C_{r',s'}$ if $r + s = r' + s' = n$ and $r - s \equiv r' - s' (\text{mod } 8)$ for such forms, thus proving Watson's conjecture completely.

2.4 Positive Values of Non-Homogeneous Indefinite Quadratic Forms

Since indefinite quadratic forms take both positive and negative values, one can consider one sided inequalities as well. Blaney [16] (1948) showed that there exist numbers Γ depending only on r, s such that if $Q(x_1, \cdots, x_n)$ is a real indefinite quadratic form of the type (r, s) and determinant $D \neq 0$, then given any real numbers $c_1, c_2, \cdots, c_n$ there exist integers $x_1, \cdots, x_n$ satisfying

$$0 < Q(x_1 + c_1, x_2 + c_2, \cdots, x_n + c_n) \leq (\Gamma|D|)^{1/n}. \tag{2.6}$$

Let $\Gamma_{r,s}$ denote the infimum of all such Γ for which (2.6) has a solution. The problem is to evaluate $\Gamma_{r,s}$ for different r, s and to determine all those quadratic forms Q and $c_1, c_2, \cdots, c_n$ for which equality is needed in (2.6) with $\Gamma = \Gamma_{r,s}$. The following table gives history of $\Gamma_{r,s}$.

$\Gamma_{1,1} = 4$	Davenport and Heilbronn [25](1947)
$\Gamma_{2,1} = 4$	Blaney 17(1950) and Barnes [12](1961)
$\Gamma_{1,2} = 8$	Dumir [26](1967)
$\Gamma_{3,1} = \frac{16}{3}$	Dumir [27](1968)
$\Gamma_{2,2} = 16$	Dumir [28](1968)
$\Gamma_{1,3} = 16$	Dumir and Hans- Gill [29](1981)
$\Gamma_{3,2} = 16$	Hans-Gill and Raka [39](1980)
$\Gamma_{4,1} = 8$	Hans- Gill and Raka [41](1981)
$\Gamma_{2,3} = (7/4)^5$	Bambah, Dumir and Hans- Gill [7](1984).

In a series of papers Bambah, Dumir and Hans-Gill [5], [6], [7] (1981, 1983, 1984) proved that $\Gamma_{r,s} = \frac{2^n}{|\sigma|+1}$ for $n \geq 6$, $-1 \leq \sigma \leq 3$.

It is conjectured by Bambah, Dumir and Hans-Gill [5] (1981) that for $n \geq 6$

$$\Gamma_{r,s} = \begin{cases} \frac{2^n}{|\sigma|+1} & \text{if} \quad |\sigma| \leq 3 \\ \frac{2^n}{\sigma} & \text{if} \quad \sigma = 4 \end{cases}$$

with possibly some exceptions.

Mary Flahive [35] (1988) proved the conjecture of Bambah, Dumir and Hans-Gill for $n \geq 21$. Aggarwal and Gupta [1], [2], [3] (1988, 1991a, 1991b) determined $\Gamma_{r,s}$ for $\sigma = -2$ and $n \geq 8$, for $\sigma = -3$ and $n \geq 9$, for $\sigma = 4$ and $n \geq 6$ and confirmed the conjecture of Bambah et al. Dumir, Hans-Gill and Woods [33] (1994) proved that

$$\Gamma_{r,s} = \Gamma_{r',s'} \text{ if } r+s = r'+s' = n, r-s \equiv r'-s' (\text{mod } 8).$$

Thus $\Gamma_{r,s}$ was determined for all (r,s) except for $\Gamma_{2,5}$, $\Gamma_{2,4}$ and $\Gamma_{1,4}$. $\Gamma_{2,5} = 32$ was proved by Dumir and Sehmi [30] (1994); $\Gamma_{2,4} = \frac{64}{3}$ was obtained by Dumir, Hans-Gill and Sehmi [32] (1995).

This leaves open only $\Gamma_{1,4}$. It is conjectured that $\Gamma_{1,4} = 8$. From the work of Jackson [45] (1971) it follows that $\Gamma_{1,4} \leq 32$. If $Q = x_1x_2 - \frac{1}{4}(x_2^2 + x_3^2 + x_4^2 + x_5^2)$ and $(c_1, c_2, \cdots, c_5) = (\frac{1}{2}, \cdots, \frac{1}{2})$, the inequality (1.1) is not soluble for $\Gamma_{1,4} < 8$. Thus $\Gamma_{1,4} \geq 8$. Dumir and Sehmi 31 (1994) proved that $\Gamma_{1,4} < 16$; Raka and Rani [61] (1997) improved this to $\Gamma_{1,4} < 12$. Recently I, with my students Swati Bhardwaj and Leetika Kathuria [13] (2024) proved that $\Gamma_{1,4} = 8$ when (i) $c_2 \not\equiv 0 \pmod 1$, (ii) $c_2 \equiv 0 \pmod 1$ and $a \geq \frac{1}{2}$, where a is the minimum of positive definite ternary quadratic forms of determinant $4|D|$. Here we obtained many critical forms for which the constant 8 is attained. For $c_2 \equiv 0 \pmod 1$, and $a < \frac{1}{2}$, many cases arise and in most of these cases we have proved that (1.1) is soluble with $\Gamma_{1,4} = 8$ without getting any critical form for which equality is necessary. But in the last some extremely difficult cases we could only improve the upper bound to $\Gamma_{1,4} < \frac{32}{3}$.

2.5 Other contributions

Chalk [18] (1947) showed that every lattice Λ is a covering lattice for

$$x_1 x_2 \cdots x_{n-1} x_n \leq d(\Lambda), \; x_i > 0 \text{ for } 1 \leq i \leq n.$$

Cole [21] (1952) showed that every lattice Λ is a covering lattice for

$$x_1 \cdots x_{n-1} |x_n| \leq \frac{1}{2} d(\Lambda), \; x_j > 0 \text{ for } 1 \leq j \leq n-1.$$

Conjecture : For $0 \leq r \leq n$, every lattice Λ is a covering lattice for the region

$$x_1 \cdots x_{n-r} \, |x_{n-r+1} \cdots x_n| \leq 2^{-r} d(\Lambda), \; x_j > 0 \text{ for } 1 \leq j \leq n-r.$$

For $n = 3, \; r = 2$, this conjecture has been proved by Bambah and Woods [10] (1977). Thus this Conjecture is completely proved for $n = 2, 3$. For $n \geq 4$, it is open for all cases other than those mentioned earlier.

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3. Minkowski's Successive Minima II

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Dedicated to the memory of Professor R.P. Bambah

In 1965, Professor R. P. Bambah presented, together with A. Woods and H. Zassenhaus [2], three different proofs of Minkowski's second inequality, one of the flagship results in Geometry of Numbers:

$$\mathrm{vol}(K) \leq \det(\Lambda) \prod_{i=1}^{n} \frac{2}{\lambda_i(K,\Lambda)}. \tag{3.1}$$

Here K is an o-symmetric convex body in $\mathbb{R}^n$ with volume $\mathrm{vol}(K)$, $\Lambda \subset \mathbb{R}^n$ is an n-dimensional lattice with determinant $\det(\Lambda)$ and $\lambda_i(K,\Lambda)$, $1 \leq i \leq n$, are Minkowski's successive minima, i.e.,

$$\lambda_i(K,\Lambda) = \min\{\lambda > 0 : \dim(\lambda\, K \cap \Lambda) \geq i\}.$$

For detailed information and definitions from Convex Geometry and Geometry of Numbers we refer the reader to the standard literature [4], [7], [8], [9]. In particular, [9[Section 6.4]] gives a detailed account to Prof. R.P. Bambah's beautiful proof as one of the three proofs in [2]. According to the three authors, the motivation for the three proofs was the search for possible generalizations of (3.1). More precisely, they wrote "Professor Mahler, during a seminar at Notre Dame University, suggested to the authors that it would be worthwhile to reexamine these proofs with a view to possible generalisation."

Forty years later, in 2005, on the occasion of R. P. Bambah's 80th birthday, the present authors wrote a short survey on generalizations and extensions of (3.1). It was published in [10] under the title "Minkowski's successive minima". The article appeared in the proceedings of the International Conference held in honor of Professor R. P. Bambah at Panjab University, Chandigarh, in 2005, where we also had the good fortune and honor to meet Professor Bambah in person. We still remember his impressive presence, which combined modesty with exceptional insight and distinction.

Now, another 20 years later, we are honored to provide a brief update of that survey in memory of Professor R.P. Bambah. We will largely restrict ourselves to a discrete analogue of (3.1), which was also a focus of [10]; for more information on classical and recent generalizations and extensions, see [1].

In 1993, the authors, together with the late U. Betke [3], studied and conjectured the following lattice-point analogue of (3.1):

$$\mathrm{G}(K,\Lambda) \leq \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} + 1 \right\rfloor. \tag{3.2}$$

Here $\mathrm{G}(K,\Lambda) = \#(K \cap \Lambda)$ denotes the lattice point enumerator. Moreover, if we denote by $\overline{K} = K - K$ the difference body of an arbitrary-not necessarily o-symmetric-convex body $K \subset \mathbb{R}^n$, then (3.1) also extends to arbitrary convex bodies in the form

$$\mathrm{vol}(K) \leq \det(\Lambda) \prod_{i=1}^{n} \frac{1}{\lambda_i(\overline{K},\Lambda)}. \tag{3.3}$$

In the same spirit, (3.2) was studied for this broader class of convex bodies, first by R.D. Malikiosis; the conjectured bound for the lattice point enumerator is as follows.

Conjecture 1 (Betke et al. [3]; Malikiosis [11])**.** *Let $K \subset \mathbb{R}^n$ be a convex body and $\Lambda \subset \mathbb{R}^n$ be a lattice. Then*

$$\mathrm{G}(K,\Lambda) \leq \prod_{i=1}^{n} \left\lfloor \frac{1}{\lambda_i(\overline{K},\Lambda)} + 1 \right\rfloor .$$

A box $[-l_1,l_1] \times \cdots \times [-l_n,l_n]$, $l_i \in \mathbb{N}_{\geq 1}$, together with the integral lattice $\mathbb{Z}^n$ shows that the bound would be tight.

In [11], Malikiosis proved Conjecture 1 for $n \leq 3$, and more generally he showed that

$$\mathrm{G}(K,\Lambda) \leq \frac{4}{\mathrm{e}}\sqrt{3}^{n-1} \prod_{i=1}^{n} \left\lfloor \frac{1}{\lambda_i(\overline{K},\Lambda)} + 1 \right\rfloor . \tag{3.4}$$

Here $\sqrt{3}$ can be replaced by $\sqrt[3]{40/9}$ if $K = -K$. Moreover, in [12] he verified the conjecture for ellipsoids in every dimension.

Recently, Tointon [13] presented a different type of upper bound on $\mathrm{G}(K,\Lambda)$ in terms of the successive minima:

$$\mathrm{G}(K,\Lambda) \leq \left(1 + \lambda_k(\overline{K},\Lambda)\right) \frac{1}{\lambda_1(\overline{K},\Lambda)} \cdots \frac{1}{\lambda_k(\overline{K},\Lambda)},$$

where the index k is chosen such that $k = \max\{j : \lambda_j(\overline{K},\Lambda) \leq 1\}$, which can be replaced in the symmetric setting by $k = \max\{j : \lambda_j(\overline{K},\Lambda) \leq 1/2\}$. Tointon's inequality improves on (3.4) in the symmetric case, as well as in various cases for general $K \in \mathcal{K}^n$. Moreover, it has, as does Conjecture 1, the nice and important feature that it implies the continuous case, i.e., the inequality (3.3). The reason is that, roughly speaking, for "fat'' convex bodies there is almost no difference between $\mathrm{vol}(K)$ and $\mathrm{G}(K)$; more precisely, Jordan measurability of convex bodies yields

$$\lim_{\rho\to\infty} \frac{\mathrm{vol}(\rho K)}{\det(\Lambda)\,\mathrm{G}(\rho K,\Lambda)} = 1. \tag{3.5}$$

In order to control the gap between $\mathrm{vol}(K)$ and $\mathrm{G}(K)$ for "thin" convex bodies, Betke et al. also started to study bounds on $\mathrm{G}(K)/\mathrm{vol}(K)$ in terms of the successive minima. And here the following inequalities could/should be true:

Conjecture 2 (Betke et al., 1993)**.** *Let $K \in \mathcal{K}^n$, $\dim(K) = n$ and $\Lambda \in \mathcal{L}^n$. Then*

$$\prod_{i=1}^{n} \left(1 - i\,\lambda_i(\overline{K},\Lambda)\right) \leq \frac{\mathrm{G}(K,\Lambda)}{\mathrm{vol}(K)} \det(\Lambda) \leq \prod_{i=1}^{n} \left(1 + i\,\lambda_i(\overline{K},\Lambda)\right),$$

where for the lower bound $n\,\lambda_n(\overline{K},\Lambda) \geq 1$ is assumed and $\mathrm{G}(K)$ might be replaced by $\mathrm{G}(\int(K))$.

Actually, in [3[Conjecture 2.2]] Betke et al. state only a conjecture about a corresponding lower bound for symmetric convex bodies in which $i\,\lambda_i(\overline{K},\Lambda)$ can be replaced by $2\lambda_i(\overline{K},\Lambda)$, and they pose the problem to consider also upper bounds (see [6[Conjecture 2]]).

The bounds in Conjecture 2 would be tight as, e.g., positive integral multiples of the standard simplex $\mathrm{conv}\{0, e_1, \ldots, e_n\}$ show (see [6]). Lucas and Freyer verified in [6] the upper bound in Conjecture 2 in the plane, and the lower bound for planar lattice polytopes. In arbitrary dimensions they proved the following weaker inequalities

$$\prod_{i=1}^{n} \left(1 - n\,\lambda_i(\overline{K},\Lambda)\right) \leq \frac{\mathrm{G}(K,\Lambda)}{\mathrm{vol}(K)} \det(\Lambda) \leq \prod_{i=1}^{n} \left(1 + n\,\lambda_i(\overline{K},\Lambda)\right). \tag{3.6}$$

One of the main ingredient of their proof is the following very useful and nice inequality [6[Proposition 1.6]]

$$(1 - \mu(K,\Lambda))^n \frac{\mathrm{vol}(K)}{\det \Lambda} \leq \mathrm{G}(K,\Lambda) \leq \frac{\mathrm{vol}(K)}{\det \Lambda}(1 + \mu(K,\Lambda))^n.$$

Here $\mu(K) = \min\{\mu > 0 : \Lambda + \mu K = \mathbb{R}^n\}$ is the covering radius of K. The lower bound, for which we have to assume $\mu(K, \Lambda) \leq 1$, has been independently proven by Dadush [5[Lemma 7.4.1]]). Observe that the upper bound in (3.6) together with Minkowski's upper bound (3.3) gives [6]

$$\mathrm{G}(K, \Lambda) \leq \prod_{i=1}^{n} \left(\frac{1}{\lambda_i(\overline{K}, \Lambda)} + n \right),$$

which in turn via (3.5) implies Minkowski's upper bound (3.3).

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□□□

4. Living with a "Legend"

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This title alludes to a tribute to R. P. Bambah, my father, published in Current Science by Professor Emeritus Rajinder Jit Hans-Gill and Professor Sudesh Kaur Khanduja, which described him as a "living legend" of Indian mathematics [9]. He met the description of himself as "legendary" with characteristic amusement and rejected it, preferring to reserve such a distinction for mathematicians of the caliber of Ramanujan, Chowla, and Harish-Chandra. In his own eyes, he was simply fortunate - fortunate to have delved into the mathematics of Ramanujan, to have been mentored by Chowla, and to have witnessed the extraordinary talent of Harish-Chandra firsthand. Between 1945 and 1947, he and Professor Chowla co-authored many significant papers investigating the congruence properties of Ramanujan's tau function [11-5]. This prolific output occurred during one of the most turbulent chapters in Indian history. While they worked in Lahore, the impending Partition and the total uprooting of their families loomed over them. Amidst this chaos, their shared devotion to mathematics became a necessary sanctuary. It was also during these years that my father forged a lifelong friendship with Professor Abdus Salam, who was a year his junior and similarly being mentored by Chowla.

His academic brilliance during this time was perhaps best captured by his Master's examination, where he achieved a perfect score of 600/600. The exam consisted of eight questions with instructions to answer only six; he solved all eight and coolly noted on his paper, "Mark any six."

By 1947, the family had been displaced to the Kingsway refugee camp in Delhi. In this hour of need, Professors F. C. Aulakh and D. S. Kothari stepped in, offering him a position teaching mathematical methods to physics students. This role proved a lifeline, providing university housing and much-needed stability. Professor Aulakh's interest in the theory of partitions in statistical mechanics had led him to my father, on Professor Chowla's recommendation. Ultimately, it was Professor Kothari who recommended my father for the 1851 Exhibition Fellowship, paving the way for him to pursue his Ph.D. at Cambridge. I believe the respect my father developed for theoretical physicists-beginning with his friendship with Salam and reinforced by the generosity of the Delhi University Physics Department-may have subconsciously influenced my own decision to pursue a career in physics.

He wished only to be remembered for the joy mathematics brought him and for his dedication to passing that passion on to his students. Throughout his life, he emphasized that rigor and persistence lay at the heart of mathematical research-a commitment he maintained even through illness and near-total blindness. Even then, his engagement with arithmetic problems and complex mathematical ideas never wavered. He was fond of quoting his mentor, Prof. Harald Davenport, with whom he did his postdoctoral work at University College London in 1952. Davenport remarked: "Mathematicians are extremely lucky. They are paid for doing what they would, by nature, do anyway. "

After losing his sight in his final years, he often asked me to look up what emerging artificial-intelligence tools were saying about recent advances in the geometry of numbers. As we explored these topics together, he would share anecdotes, some of which I have recounted here. He remained mildly sceptical of technology, feeling that Google and ChatGPT offered, at best, a curated collection of facts. For him, true knowledge was never passive accumulation, but a deeply transformative force. As he once said, "Knowledge is not something to be packed away in some corner of our brain, but something that enters into our being, colours the emotion, haunts our soul, and is as close to us as life itself." He viewed such internalized knowledge as an "overmastering power which, through the intellect, moulds the whole personality, trains the emotion, and disciplines the will," and he held a lifelong reverence for the rigor, structure, and logical order required to establish mathematical truth.

For this reason, he consistently urged me never to abandon physics or mathematics, and to continue learning and working even after retirement. He often remarked that, had he known he would live so long, he would have pursued new areas of mathematics more actively; he regretted that his knowledge was not broad enough to fully absorb some of the major breakthroughs of his later years-most notably Andrew Wiles's proof of Fermat's Last Theorem and the deep connections it established between elliptic curves and modular forms. He encouraged me to study the proof to uncover its underlying ideas and expressed a particular wish that he had better understood the Taniyama-Shimura-Weil conjecture on which Wiles's work was founded.

Another landmark proof he witnessed was Thomas Hales's proof of Kepler's conjecture, which he felt was more closely aligned with his own work, as it related to the dual problem of sphere packing-the covering problem. While packing concerns arranging spheres without overlap, covering seeks the minimum density required to ensure that every point in space lies within at least one sphere.

During his final days, he took me through his lifelong mathematical journey, tracing a path from his first encounter with the geometry of numbers to its present state-a landscape we navigated together using modern web tools that continued to intrigue him even at the age of ninety-nine. As this journey began in 1950 and spanned seventy-five years, I can offer only highlights here. He earned his Ph.D. from St. John's College, Cambridge, in 1950, under the supervision of Louis J. Mordell. The story was that he went to Mordell to ask for a problem; Mordell gave him a difficult one, which my father solved in six months, much to Mordell's surprise. Mordell declared this sufficient for a Ph.D., and my father completed it in just one and a half years. This marked the beginning of his work on lattice coverings by spheres-and more broadly, convex bodies-in Euclidean spaces, focusing on how to cover all of with spheres centered at lattice points while minimizing covering density.

From 1952 to 1954, my father was at the Institute of Advanced Study at Princeton. There he interacted with Prof. Atle Selberg. Before going to Princeton, his friend Abdus Salam gave him a piece of work that he had done on Longitudinal photons . At that time, Salam was still early in his career, and his ideas were not yet widely known. It was through his close friendship with my father forged during their years under Chowla's mentorship, that this work found an unexpected conduit. My father, though a mathematician by training, had a deep interest in theoretical physics and a strong intuitive grasp of its conceptual structure. Recognizing the importance of Salam's argument, he conveyed the essence of the work to Freeman Dyson, who was then one of the central figures shaping the modern formulation of quantum electrodynamics. Dyson immediately grasped the significance of the result. The clarification of the role of longitudinal photons fit naturally into the emerging framework he was developing. Dyson was surprised and told my father "I said it could be done I am surprised that it would be done." The episode exemplified a recurring pattern in my father's life: though he rarely sought the spotlight, he served as a quiet but effective bridge between ideas, disciplines, and people. It also reflected the intellectual generosity of that generation, in which important insights often travelled informally, carried by trust, shared understanding, and deep respect for ideas rather than by priority or publicity.

On the way back from Princeton, via England, my father travelled by ship. On that ship was a recently qualified young doctor named Saudamini Parija. She had completed her M.R.C.P. in Edinburgh and, after a stint as a resident at Hammersmith Hospital, London, was returning to her home in Cuttack, Orissa. The two became acquainted during the voyage, and she went on to become my mother.

It was a serendipitous match across states, cultures, and castes-revolutionary for its time-and destined for fifty-five years of togetherness. My mother complemented my father's personality with her generous spirit, social acumen, and deep sense of charity. She was oblivious to distinctions of class; to her, everyone was equal, and she served all as a doctor. Together they were a power couple-dignified and gracious-who opened their home to everyone: students, colleagues, the destitute, and dignitaries alike.

The last fourteen years of my father's life, lived without my mother, were a half-hearted existence. My father had a long association with Ohio State University, where he was invited to join as a professor in 1964 by Prof. Arnold Ross. Ross had founded the celebrated Ross Program, an intensive six-week summer residential program for motivated pre-college students focused on deep, proof-based number theory. As a faculty member at OSU during the years the program was based there, my father was part of the distinguished mathematical community that nurtured its success. The program has shaped many academic careers in mathematics, including those of several Indian mathematicians. I too had the honour of attending the program and later teaching in it as a counsellor at the University of Chicago, when the Ross Program relocated there briefly in the 1970s. Its guiding philosophy-"think deeply about simple things"-resonated deeply with him.

He took particular joy in teaching in the program, especially alongside his close friend Alan Woods, with whom he co-authored approximately twelve papers at Ohio State University. He valued collaboration deeply and took pride in his work with H. Davenport, K. F. Roth, C. A. Rogers, and N. Sloane. Together with Zassenhaus and Rogers, he bridged number theory, convex geometry, and algebra in their paper On lattice constants [6], which at the time provided some of the tightest known bounds for lattice covering problems in low dimensions and anticipated later developments in algebraic lattices and automorphic forms.

In the 1970s, he became deeply engaged with Minkowski's conjecture. While Minkowski proved the conjecture for two dimensions, its difficulty increases rapidly with dimension. One of my father's most significant achievements was proving the conjecture for five dimensions, jointly with Alan Woods [7]. Beyond this, he worked with his students-including V. C. Dumir and R. J. Hans-Gill-to establish bounds and estimates in higher dimensions, narrowing the terrain for future proofs.

In 1982, he resolved a problem concerning the covering radius of the Leech lattice in twenty-four dimensions [8], a paper that famously grew out of an informal dinner conversation at a conference. Decades later, he was delighted by Maryna Viazovska's 2016 proof that the Leech lattice yields the densest possible sphere packing in that dimension. Even in his last days, he asked me to play recordings of her lectures so that he could follow her ideas.

These mathematical accomplishments ran parallel to his deep commitment to higher education in India. In 1969, the VC of Panjab University, Shri Suraj Bhan, came especially to Ohio State University to ask Prof. Bambah to return to Panjab University., and nurture the Mathematics department. He returned to India and under his leadership, Panjab University established India's first Centre for Advanced Study in Mathematics, significantly enhancing its international academic standing. As Vice-Chancellor of Panjab University from 1985 to 1991, he provided steady and visionary leadership during a period of intense unrest. Despite widespread fear and instability, he ensured that the campus remained a sanctuary for intellectual pursuit. Even after retirement, he continued to serve as a Professor Emeritus and long-standing Senate member, embodying a model of academic leadership grounded in integrity and humility.

Throughout his life, he emphasized one principle above all: that research and teaching must go hand in hand, for there can be no genuine education without original thinking. He believed a university should be a place where students are not merely taught, but encouraged to think independently, rigorously, and courageously.

As I reflect on his journey, I am reminded that his life was shaped not only by the theorems he proved or the conjectures he resolved, but by a profound sense of wonder and a relentless pursuit of clarity. Even as his physical vision faded, his intellectual vision remained undimmed, always directed toward new horizons of mathematical truth. He leaves behind a legacy of integrity, brilliance, and humility, and a deep conviction that knowledge-once truly internalized-has the power to transform not only the mind, but the very soul of a person.

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The family at
Columbus, Ohio 1967

President, Indian Science Congress,
Ranchi, with Prime Minister Indira
Gandhi, 1984

Receiving the Padma Bhushan from
President R. Venkataraman, 1988

Donning the Honorary Doctorate from
Panjab University, 2014

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5. A Tribute to Professor R. P. Bambah

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Professor Ram Prakash Bambah (30 September 1925 - 26 May 2025) was an eminent mathematician, a dedicated teacher, an able administrator and above all a great human being. I had heard about his extraordinary brilliance when I was studying for my B.A. degree in Dev Samaj College for Girls, Ambala City, during 1968-1971. This college, originally established at Lahore and relocated to Ambala City after the partition of India, was affiliated to Panjab University until mid 1974. The Vice Principal of that college, Professor Ram Yash, was a fan of Professor Bambah. In fact, they studied together at Government College, Lahore around 1945. Accompanied by Professor Ram Yash and his wife, I met Professor Bambah for the first time in his office in the Department of Mathematics, Panjab University Chandigarh in June 1971. I had planned to seek admission in for M.A. in this Department, which was being headed by Professor Bambah at that time. "You are joining the Department as a topper of Panjab University, you should work hard to retain this position" were his first words to me. I also partly remember his address given to the outgoing batch of M.A. students at the annual function of the Department in March 1972. He said "Mathematics is a difficult subject. You have become hardworking in trying to understand it well. So you will succeed wherever you go". Our batch was not lucky enough to be taught by him. But I came in close contact with him when I joined the Ph.D. program in this department from September 1973 and later as a faculty here in 1978. During the session 1974-75, Professor Bambah gave a one-year course on Algebraic Number Theory to postgraduates and Ph.D. students. My long-time colleague Madhu Raka, who later worked on several problems raised by Professor Bambah, and I attended this course. Bambah Sir used to take our class from 8 am to 9 am daily for six days a week. As a teacher, he was simply extraordinary. He would always be on time for our class and would spend the first few minutes recalling what he covered in the previous lecture. He used to fix an extra class in advance if he had to go out of station for some official work. One remarkable thing is that I always found him full of energy ready to discuss Mathematics. I fell in love with the subject he taught. His teaching played a great role in shaping my research career, as I wrote my Ph. D. thesis under the supervision of another luminary of the Department, Professor I. S. Luthar, on advanced Algebraic Number theory, and later supervised a dozen Ph.D. theses related to this topic.

While working as a faculty in the Department, Professor Bambah used to inspire his younger colleagues a lot. He would keep track of our achievements and felt happy when we accomplished something worthwhile. Had Professor Bambah not encouraged me so strongly, my research career would have met an early end. After the birth of my elder son in 1980, I got too absorbed in his care and became a little slack in research. As a result, I could not write any research paper for a couple of years. My husband who during those days was an Assistant Professor in Post Graduate Institute of Medical Education and Research (PGIMER), Chandigarh was not affected by the arrival of our child and was actively engaged in research as before. Professor Bambah, being the then Vice Chancellor of Panjab University, was on the Governing body of PGIMER, and had access to its annual report. After looking at one of these reports, while sipping tea in the lawns of the Physics Department of our university with younger colleagues including Madhu Raka and myself, he said "Sudesh, your husband has published four papers in one year whereas you have not published even one paper during the last four years!". This incident proved a turning point in my career. I decided there and then that I will not lag behind my husband and promised Bambah Sir that I shall work hard to come up to his expectations. Of course God helped me to fulfil my promise to the extent that on 21st March 2023, when I delivered the Srinivasa Ramanujan Medal Lecture in IISER Mohali, Professor Bambah who could not attend the lecture physically due to health issues, got emotional after listening to it online and conveyed to the audience through his daughter Professor Bindu Bambah that he is proud of Sudesh.

Besides being super intelligent, Professor Bambah was extremely hard working. His Ph.D. advisor, L. J. Mordell at the University of Cambridge, gave him a research problem in January

1949 and he was able to solve it by April 1949. He got his Ph.D. in 1950. He put his heart and soul in anything he took up, whether it be teaching or research or university administration. He was the Vice Chancellor of Panjab University for almost seven years during 1984 to 1991. It may be pointed out that this period was one of the most traumatic and turbulent chapters in the history of Punjab State, defined by the rise of a militant insurgency. Following the Indian Army's Operation Blue Star in June 1984, the state was consumed by a cycle of violence that saw thousands killed, including civilians, militants and security personnel. But under the able vice chancellorship of Professor Bambah, Panjab University worked smoothly.

Celebration of Prof. Bambah's 100th birthday (posthumous), with a cake in the shape of a book by his daughters, and another cake baked by the author.

Besides a sharp brain, Professor Bambah was blessed with a fantastic memory. He enjoyed this blessing till the last day of his life. His YouTube interview giving vivid details of his association during his young days with former Prime Minister Dr. Manmohan Singh after latter's demise on December 26, 2024 bears testimony to this memory. In the summer of 2021, I went to Professor Bambah's house to show him the draft of my book entitled "A Textbook of Algebraic Number Theory", which is dedicated to him. After reading the introduction and the table of contents, he flipped through its pages for about half an hour, he told me that the third chapter of the book should contain a reference to a paper from 1964 whose single author was his former Ph.D. student Gurnam Kaur. Some results from Gurnam Kaur's thesis were on the tip of his tongue. I was amazed by his remarkable memory and deep interest in Mathematics. During his last meeting with me in April 2025, he asked me about the well-being of my youngest grandson in USA, remembering my daughter-in-law Neha by name. Such was his attachment to his former students and their families. His last words to me were "Never give up Mathematics till the end". Only time will tell how faithfully I follow his advice.

In their illuminating article "Century of the Punjab School of Mathematic" (published in the February 2026 issue of Panjab University Research Journal), Professor Arun Grover, former Vice Chancellor, PU, and Dr. Surinder Singh Kainth, Head, Department of Mathematics, PU write "It is a moment of great reflection that the centenary of the Punjab School of Mathematics coincides with the birth centenary of one of India's most distinguished mathematicians, Professor Ram Prakash Bambah (September 30, 1925 - May 26, 2025). His passing away earlier this year lends a solemn dimension to the celebrations, providing an occasion not only to honour a century of mathematical excellence at Panjab University[1] but also to commemorate the life and legacy of Professor Bambah..."

Albert Einstein said of Mahatma Gandhi "Generations to come will scarcely believe such a one as this ever in flesh and blood walked upon this earth". I and many other students of Professor Bambah also feel the same about Professor Bambah. My last offering to him was his favourite coffee walnut cake baked by me when his daughters Bindu and Sucharu invited me to their father's residence while celebrating his 100^{th} birthday, alas, posthumously.

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[1]The name and location of this university have undergone various changes throughout its history, most notably as the University of Punjab at Lahore and after partition as Panjab University, Chandigarh.

6. Professor R. P. Bambah as my Mentor, Vice Chancellor and Senator of Panjab University, Chandigarh: A Reminiscence

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Introduction

Ram Prakash Bambah (RPB) was an alumnus of Government College, Lahore-the nucleus of Panjab University (PU) in pre-Independence India. He began his research under the legendary Prof. Sarvadaman Chowla (SC) at Lahore, shortly before India's Independence. Subsequently, he was appointed on a leave vacancy to teach Mathematical Physics in the Department of Physics at the University of Delhi (DU) by Prof. D. S. Kothari, a contemporary of SC at Cambridge University. This vacancy arose when Dr. F. C. Auluck (FCA), also a student of Chowla at Lahore, proceeded on a three-year Research Fellowship funded by the National Institute of Sciences (erstwhile INSA). In 1950, RPB proceeded to Cambridge University for his PhD. Upon his return, he was appointed as Reader in Mathematics by Panjab University at Government College, Hoshiarpur, which had been adopted as a constituent college, while the main PU campus was being developed in the newly planned capital city of East Punjab, Chandigarh. RPB later rose to become Head of the Department of Mathematics at Chandigarh and was selected as the fifth full-time Vice-Chancellor of Panjab University in 1985-a particularly turbulent period in Punjab's history.

In 1990, during his tenure as Vice-Chancellor, RPB inducted me as Professor of Physics at Panjab University. Like RPB, I completed my Bachelor's and Master's degrees (1968–1972) at PU before proceeding to the Tata Institute of Fundamental Research (TIFR) for my PhD in Physics, where I remained until 2012. TIFR granted me lien to join PU in February 1991, and my three-year tenure (1991–1994) as a faculty member at PU proved pivotal in my subsequent selection as the tenth full-time Vice-Chancellor of Panjab University in 2012.

RPB served as a nominated member of the Panjab University Senate for three consecutive terms of four years each from 2008 to 2020. Over the years, I have had the privilege of interacting with him in multiple capacities-as a student, a faculty colleague, and a fellow Senator.

My student days at Physics Honours School, Panjab University, Chandigarh

My earliest recollection of Ram Prakash Bambah (RPB) dates back to 1968, when I was a first-year student in the Physics Honours School at Panjab University. His official residence was located opposite the home of one of my classmates, whose father was employed in the Administrative Office of PU. Although we only saw him occasionally from a distance, my classmate was well acquainted with his family. Among physics students, RPB's persona and academic stature inspired a sense of awe and reverence. He was known for taking the early morning class at 8:30 a.m. for Mathematics Honours and M.A. students, after which he devoted himself to research. Regrettably, during that period, senior faculty from the Mathematics Department did not teach physics students who pursued mathematics as a subsidiary subject for two years.

In 1970, RPB was appointed Dean of University Instruction (DUI) at the PU campus-a position unique to Panjab University since 1922. Traditionally, the senior-most Professor is appointed as DUI for a two-year term, performing a role comparable to that of a Pro–Vice-Chancellor in other Indian universities.

During my student years at PU (1968–1972), approximately a dozen National Science Talent Search (NSTS) scholars were enrolled across various science departments. I took the initiative to organize interaction sessions between these NSTS scholars and the Heads of Departments that had been recognized as University Grants Commission Centres of Advanced Study (CAS) in Botany, Chemistry, and Mathematics. The then Heads of the Botany and Chemistry Departments had been students of Shiv Ram Kashyap and Shanti Swarup Bhatnagar, respectively, at Lahore. Listening to their anecdotal recollections of academic life in Lahore was both inspiring and deeply evocative.

My induction as Professor of Physics at PU by Vice Chancellor Prof. Bambah

Prof. Ram Prakash Bambah assumed office as Vice-Chancellor of Panjab University on 1 January 1985, at a time when Punjab was under President's Rule. An elected government led by Mr. Surjit Singh Barnala came to power in September 1985 and remained in office until June 1987, after which President's Rule was once again imposed in the state. Consequently, RPB's tenure as Vice-Chancellor was marked by severe administrative and political constraints.

I was inducted as Professor of Physics during his third three-year term as Vice-Chancellor, at a time when there were overt pressures on him to restrict faculty appointments to candidates belonging to the majority community in Punjab. RPB did not yield to such diktats. He was, perhaps, acutely conscious of the fact that he himself had been denied consideration for a teaching position at Lahore in pre-Independence India due to an undeclared moratorium on appointing non-Islamic candidates to government posts in the then Punjab.

My interview for the Professorship at Panjab University in August 1990 was conducted away from the Vice-Chancellor's office premises. Nevertheless, RPB ensured that my selection was endorsed by the University Syndicate the very next day, and the Registrar issued my appointment letter without delay. I was granted six months to join the post. When I arrived in February 1991, I requested an additional two months to complete experiments underway at TIFR. Instead of extending the joining period, RPB exercised his discretionary powers as Vice-Chancellor-on the recommendation of the Department Chairperson-and granted me two months of duty leave immediately after joining.

I also sought either out-of-turn residential accommodation or consideration of my service at TIFR for advancement in the university housing queue. While RPB was sympathetic, he expressed his inability to accede to these requests due to procedural constraints. However, he successfully obtained approval from the Syndicate to grant me a higher starting salary to partially offset the cost of renting accommodation outside the campus. I accepted these arrangements and returned to Panjab University, Chandigarh, in May 1991 to formally begin my duties in the Department of Physics.

During this intervening period, RPB's stay in Chandigarh became increasingly difficult due to unwarranted threats from militant groups. He was advised to proceed on leave during the summer break and eventually resigned as Vice-Chancellor while abroad. His successor formally assumed office at Chandigarh on 22 July 1991.

It is noteworthy that Prof. Bambah's own requests for study leave and duty leave had been handled with considerable generosity by three successive Vice-Chancellors of Panjab University between 1952 and 1965. In turn, he remained consistently sympathetic to similar requests from faculty members during his tenure as Head of the Department of Mathematics, Dean of University Instruction, and Vice-Chancellor.

The period that followed remained deeply turbulent. Two faculty members of Panjab University lost their lives to terrorist violence during the tenure of RPB's successor. Although an elected government assumed office in Punjab in February 1992, the Chief Minister was assassinated in a suicide attack in August 1995. Amidst this instability, I received an offer to move from Panjab University to the School of Physical Sciences at Jawaharlal Nehru University, New Delhi. However, I chose instead to return to TIFR in May 1994.

Many years later, during a subsequent meeting, Prof. Bambah concurred with this decision. He remarked that my return to TIFR enabled me to further strengthen my academic credentials to a level that ultimately made me a serious contender for appointment as Vice-Chancellor of Panjab University in 2012.

Serving as member of a UGC Committee along with Prof. Bambah

While serving as Vice-Chancellor of Panjab University, Prof. Ram Prakash Bambah was appointed Chairperson of the Governing Council of the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune. In 2008, the University Grants Commission (UGC) constituted a committee under the chairmanship of Prof. Yash Pal to review two decades of functioning of its three Inter-University Centres (IUCs): IUCAA, Pune; the Inter-University Accelerator Centre (IUAC), New Delhi; and the UGC–DAE Consortium for Scientific Research (CSR), Indore.

Prof. Yash Pal asked me to liaise with the Directors of the three IUCs and, with the assistance of a Joint Secretary of the UGC, to prepare a structured background report for consideration by the Committee. When the Committee convened in New Delhi, I found myself participating alongside the Chairperson of the Committee, Prof. Bambah, and the Director of RRCAT in the review of the three IUCs. Prof. J. V. Narlikar, the founder Director of IUCAA, was also present during the first two days of the review meeting held at IUAC, New Delhi. The entire exercise proved to be an invaluable learning experience for me.

I subsequently prepared the draft of the Committee's report for submission to the Chairperson, UGC. At that time, I was unaware that Prof. Bambah and Prof. Yash Pal had known each other since their days in Lahore, where they had lived in the same locality prior to Independence. Prof. Yash Pal had also been appointed Honorary Professor at Panjab University under a provision of the original Indian Universities Act of 1904, which governed the then five universities of India located at Calcutta, Madras, Bombay, Lahore, and Allahabad.

Seeking guidance from PU Senator Prof. Bambah while serving as VC, PU

On the evening prior to my assuming office as Vice-Chancellor of Panjab University, my predecessor arranged an interaction with members of the PU Senate. During this occasion, Prof. Ram Prakash Bambah offered me counsel that left a lasting impression. He advised me not to be overawed by accounts of the achievements of earlier Vice-Chancellors, and instead urged me to "be yourself and be decisive." He added that if, in hindsight, a decision appeared to have been wrongly made, one should not hesitate to correct it. He further advised me to carry forward constructive initiatives already set in motion. In confidence, he also shared that my predecessor aspired to be appointed Professor Emeritus of Panjab University.

In turn, I shared with him the ideas I had articulated before the Search Committee for Vice-Chancellor, PU. Central among these was a proposal to commemorate 150 years of higher education in Punjab, set against the backdrop of the 150th birth anniversary of Ruchi Ram Sahni (1863–1948), a pioneer of science popularisation in the region. Prof. Bambah had known Ruchi Ram Sahni at Lahore. Sahni had mentored his son, Birbal Sahni, FRS, as well as Shanti Swarup Bhatnagar, FRS. RPB had met Bhatnagar at Delhi University and was fully aware of the seminal role played by him as the first Chairman of the UGC in initiating the construction of the Panjab University campus at Chandigarh, modelled on the residential campus of BHU-an institutional form that did not exist at Lahore.

Taking guidance from Prof. Bambah's advice, I proposed in my opening address to the Syndicate on 4 August 2012 that this body formally records its appreciation of my predecessor's efforts in initiating the construction of several new buildings, and that he be conferred the title of Professor Emeritus. While some Syndicate members were initially hesitant, both proposals were unanimously approved. This marked a meaningful beginning to my tenure.

Prof. Bambah strongly endorsed my initiative to institute an Annual Panjab University Foundation Day Lecture, with the inaugural lecture delivered by PU alumnus Prof. Romila Thapar in October 2012. He had known her for many years and was well aware of her family's deep connections with Lahore.

A new Senate for the term 2012–2016 was notified with effect from 1 November 2012. The Chancellor re-nominated Prof. Bambah, and also included Prof. S. S. Johl (then Chancellor, Central

University of Punjab), Ambassador I. S. Chadha, IFS (Retd.), Mr. Tarlochan Singh (former Chairperson, National Commission for Minorities), and Prof. D. V. S. Jain (Emeritus Professor, Department of Chemistry) in his list of other nominees in the new Senate. All of them had long familiarity with the governance structures of PU, and I could rely on their objective assessments. Prof. Bambah and Prof. Johl, in particular, assumed the role of my mentors and provided invaluable guidance as I navigated several complex challenges.

One such issue concerned the retirement policy of PU, which stipulated retirement at 60 years with the possibility of re-employment for three years. In the first meeting of the new Senate, this was enhanced to five years of re-employment on last salary drawn, subject to submission of an annual performance report acceptable to the Vice-Chancellor's office. Faculty inductions made by my predecessor during the final months of his term also required careful defence in the Senate; Prof. Bambah's guidance proved crucial in their successful handling. Several long-pending and intractable issues were referred to Bambah Committees, and their recommendations were subsequently processed through the Syndicate and Senate.

Within the PU governance framework, the Syndicate-elected annually-processes all executive decisions of the Vice-Chancellor. Litigation-stalled appointments, including that of the Controller of Examinations, were resolved within the first six months of my tenure. Although my predecessor had advertised over 250 teaching and more than 300 non-teaching posts, during my first three-year term I was able to fill nearly 100 teaching positions and almost all non-teaching posts. A full-time Registrar was appointed in October 2014 after a gap of two decades, along with regular appointments of Director of Sports, Public Relations Officer, Law Officers, and others.

On the academic front, the year-long commemoration of the 150th birth anniversary of Ruchi Ram Sahni unfolded across entire Punjab. A commemorative postal stamp was released in October 2013, and a three-day national seminar titled "Making of Modern Punjab: Education, Science and Social Change" brought together eminent scholars at the PU campus. Prof. Bambah, Prof. Yash Pal, Prof. J. V. Narlikar, Dr. Srikumar Banerjee, Prof. G. S. Grewal, Dr. Gurdial Singh, and others participated in these events.

During the RRS commemorations in 2013-14, I had reached out to the Vice-Chancellors of Panjab University Lahore and Government College University Lahore-both physicists. Dr. Mujahid Kamran, VC, PU Lahore, sent me copies of his book The Inspiring Life of Abdus Salam, which vividly described the bond between Abdus Salam and Prof. Bambah, both students of Sarvadaman Chowla at Lahore and later at Cambridge, along with Har Gobind Khorana. Presenting this book to Prof. Bambah, and listening to an account of his conversations with Salam was a deeply moving experience.

Prof. Bambah and Prof. Johl also guided the shortlisting of national and international luminaries for Honoris Causa degrees and PU Rattan Awards, which required ratification by the Syndicate, Senate, and the Chancellor. In 2013, Prof. Romila Thapar, Prof. M. S. Raghunathan, FRS, and Prof. Asoke Sen, FRS, received honorary degrees. In 2014, I proposed awarding the D.Sc. (Honoris Causa) to Prof. Bambah himself. Though initially reluctant, he graciously accepted following a unanimous appeal from the Senate. The other recipient that year was Gyanpith Awardee Dr. Gurdial Singh.

Subsequent recipients included Prof. M. S. Swaminathan, FRS, Prof. Manjul Bhargava (Fields Medalist), and Nobel Laureate Kailash Satyarthi (2015); Prof. Har Kishan Singh (2016); Prof. Murali Manohar Joshi, Dr. Nuruddin Farah, and Prof. G. S. Khush, FRS (2017); and Justice Jagdish Singh Khehar, Prof. M. M. Sharma, FRS, and Prof. Sir Tejinder Singh Virdee, FRS (2018).

PU also honours prominent alumni and national icons with its Rattan Awards during annual convocations. Dr. Nitya Anand was chosen for PU Vigyan Award (2013), and Dr. Sukhdev and Mr. F C Kohli were chosen for PU Udyog Rattan Award for the years 2013 and 2014, respectively. The awards for the 2013 and 2014 were presented during the convocation of 2014. Hosting Dr. Nitya Anand, Dr. Sukh Dev, Mr. F. C. Kohli, and Prof. Bambah-contemporaries from Lahore-together at the PU Guest House in 2014 remains among my most cherished memories.

Dr. R. P. Bambah (1925-2025), Dr. Nitya Anand (1925-2024), Dr. Sukh Dev (b: 1923) and Dr. F. C. Kohli (1924-2020) at PU Guest House in 2014.

Most awardees honoured during annual convocations from 2013 to 2018 delivered expository lectures at the PU campus, almost all of which Prof. Bambah attended along with me. On the occasion of 90th birth year of RPB, PU instituted an annual Sarvadaman Chowla Lecture, the first of which was delivered by Manjul Bhargava. RPB presided over this lecture, and recalled for us that Klaus Roth (1925-2015), his contemporary and co-author of a paper written at Cambridge, won the Fields Medal in 1958.

In recognition of the efficacy of the visits of national and international icons to PU to enrich its academic environs, Prof. Bambah brought to my attention a two decades old Senate decision to institute Chair Professorships in the names of Maharaja Ranjit Singh, Tagore, Mahatma Gandhi, Nehru and Lal Bahadur Shastri. Such Chairs had been lying vacant as suitable persons could not be enticed to accept regular positions at PU. A high powered committee was tasked to suggest a new algorithm to fill up these positions as Visiting Faculty on the lines of 'Honorary Fellow' of TIFR. A budget provision was got approved against an Endowment fund of PU. Shri Gulzar was invited to the Tagore Chair, Lt. Gen K J Singh (Retd) was invited to the Maharaja Ranjit Singh Chair, Smt. Ila Bhat to the Gandhi Chair, Yogesh K Alagh to the Nehru Chair and Kailash Satyarthi to the Shastri Chair. Their appointments were approved by the Senate, and all these icons started to visit PU campus occasionally as per their convenience.

The author with Prof. Bambah at his Chandigarh residence on his 99^{th} birthday on September 30, 2024.

My reappointment for a second three year term as Vice-Chancellor was notified on 13 April 2015. Shortly thereafter, a complaint alleging financial mismanagement over 25 years by VCs of PU was lodged with the MHRD, prompting a fact-finding inquiry. To respond effectively, I revisited the financial evolution of PU since its founding at Lahore in 1882. With Prof. Bambah's guidance, I gained deeper insight into how the financial architecture evolved as PU transformed into a residential campus at Chandigarh in 1958, with independent departments in over forty domains. PU did not have a campus of its own at Lahore, there were only about twenty faculty members in all subjects appointed by the University at Lahore. PU's tradition of meticulous record-keeping, including audited accounts and periodically updated Accounts Manual, enabled us to respond comprehensively to all queries. The revised Accounts Manual approved during my tenure further strengthened this framework, allowing the University to successfully weather the crisis.

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7. Professor R. P. Bambah: Bhishma Pitamaha of Modern Indian Mathematics

M. S. Raghunathan

I first heard of Professor Bambah and the Mathematics School at Panjab University (PU) in 1960 when I joined the Tata Institute of Fundamental Research (TIFR) as a PhD student. He was talked about as one of the few (3 or 4) outstanding mathematicians in the country outside TIFR with an excellent international reputation; and the mathematics department at Panjab University, of which he was the prime architect, as the best among the university departments in the country. I do not remember when I met him first: it was perhaps during one of his short visits to TIFR in the early sixties when I was still a student. Towards the end of the sixties, I got to know him somewhat well. At some point, he became the Chair of the Mathematics Committee of the Board of Research in Nuclear Sciences (BRNS), and I was the secretary during part of his tenure.

In 1972 both Bambah and I became members of the "Steering Committee" of the Binational Conference on Mathematical Education, and that gave me many opportunities for interaction with him. In 1983, the National Board for Higher Mathematics (NBHM) was formed, and both Professor Bambah and I were made members. That meant greater interaction with Professor Bambah over a long period. The interactions were always pleasant; he was always courteous and considerate, treating me as an equal despite his seniority in age as well as in professional status. Our mathematical interests were far from close, and our interactions had mostly to do with the promotion of mathematics in the country.

Professor M S Narasimhan quit the Chairmanship of NBHM in 1987, and Bambah played a key role in persuading the DAE Secretary to appoint me to succeed Professor Narsimhan. He was himself offered the chairmanship of NBHM, which he declined. He played a key role again in getting me appointed as Chair of the Governing Council of the Harish-Chandra Research Institute (HRI) (then Mehta Research Institute of Mathematics and Mathematical Physics), even as he continued as a member of that body.

There was one occasion when his mathematical interests came close to mine. I had observed that a conjecture of Oppenheim (which was of interest to Professor Bambah) would be a consequence of an assertion in Homogeneous Dynamics. The assertion came to be known as my conjecture and was proved by Marina Ratner. Professor Bambah invited me once to Chandigarh to give a talk on the Oppenheim Conjecture. I got acquainted with his family on that occasion.

Professor Bambah was the leading Number Theorist of his generation in India. Apart from his own research contributions, he was the mentor for many Number Theorists from different parts of the country. He built an excellent school in mathematics in Chandigarh. He contributed immensely to the development of mathematics in the country. He was the veritable Bishma Pitamaha of Indian Mathematics of modern times.

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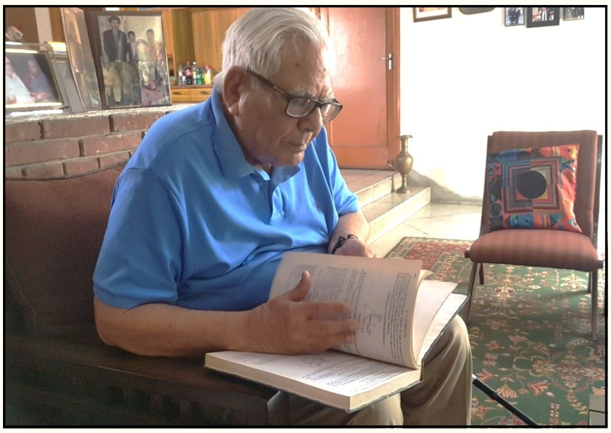
Professor Ram Prakash Bambah, in his study at age 94.



Courtesy *Bhāvanā* and Prof. C. S. Aravinda

8. Beyond Numbers: Psychological Reflections on the Life of Professor R. P. Bambah

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[1]The world of mathematics is often perceived as abstract-defined by cold logic and symbolic manipulation, detached from the human spirit. Yet behind every theorem and proof lies a mind shaped by emotion, resilience, curiosity, and a value-driven search for truth. Few lives reflect this blend of intellectual brilliance and human depth as profoundly as Professor Ram Prakash Bambah (1925-2025).

His departure on 26 May 2025, just four months shy of what would have been his centenary, was mourned across the mathematics community-not merely as the loss of a towering scholar but of a deeply humane soul. This is not just a story of academic milestones; it is also the saga of a life well-lived-marked by serene perseverance, empathetic leadership, intellectual humility, and unwavering integrity.

Born on September 30, 1925, in Jammu, Professor Bambah's early life was a constant exercise in adaptation. His father's frequent transfers across Punjab and Balochistan led young Ram Prakash continually navigating new environments, forming new social bonds, and adjusting to different schools. Psychologically, this fostered a remarkable adaptability and a quiet confidence in his ability to handle change. It is a key insight into his later career, where he moved seamlessly between different academic cultures-from Lahore to Cambridge to Princeton and back to Chandigarh.

In an interview, Prof. Bambah spoke about his intellectual inheritance, recalling that his mother had no formal education. She had only attended a Gurudwara school, where she learned to write Punjabi and perform calculations. Yet, she possessed remarkable speed and accuracy in arithmetic. "I think she could easily have been better than me," he remarked. Though not formally educated, her sharpness formed a subtle part of his intellectual inheritance, showing how natural ability can flourish even outside classrooms. His father, a Railway Guard who rose to become Chief Yard Master, instilled in him discipline and a strong sense of duty.

His family lived in various towns-Wazirabad, Sialkot, Quetta, and later Lahore. These places, connected by railways and bustling with life, formed the backdrop of his childhood. Despite frequent moves, young Bambah distinguished himself early in school. Teachers recognized in him a spark above the ordinary and he was allowed to skip third grade and he was ranked. When he took his matriculation examination, he placed ninth among 40,000 students across Punjab, an astonishing achievement that earned him a scholarship and a coveted place at Government College, Lahore.

The Government College, Lahore was a crucible of brilliance, attracting students who would go on to change the world, including future Nobel Laureates Abdus Salam and Har Gobind Khorana. The entry into this college marked a turning point in Bambah's life. Surrounded by the intellectual elite of Punjab, Bambah's initial humility was tested and refined, and it proved to be a period of intense psychological growth.

The greatest influence on the young Bambah during his time at Government College was Professor Sarvadaman Chowla, a number theorist trained under the legendary J. E. Littlewood at Cambridge. Chowla's teaching style was spontaneous and intuitive; he thought aloud, inviting students into the very process of mathematical discovery. He had a unique ability to spot and nurture exceptional talent, identifying a trio of students-Jagdish Luther, Mahendra Raj, and Ram Prakash Bambah-as the brightest among their peers.

Chowla's unconventional teaching style, where he encouraged students to engage with the process of discovery, provided a supportive space for learning. His perfect 600 out of 600 score on his M.A. exams was more than an academic achievement; it symbolized Chowla's trust in Bambah and became a psychological milestone. For Bambah, Chowla's trust unlocked possibilities that propelled him into the global mathematical arena.

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[1]Reprinted, with gratitude, from the Centennial Issue of The Indian Journal of Psychology, September 2025.

The friendships he forged with Jagdish Luther and Mahendra Raj were not mere acquaintances but lifelong bonds, spanning of over eight decades, built on mutual respect and shared struggle. Their families became intertwined, creating a powerful social support network that provided stability and emotional connection throughout their lives. This deeply rooted sense of belonging helped ground Bambah's soaring intellect, protecting him from the isolation that can sometimes accompany genius.

In late 1945, while awaiting his official M.A. results, a serendipitous opportunity arose. Professor Chowla recommended him to Hansraj Gupta at Government College, Hoshiarpur, to fill in for a faculty member on leave. Bambah went merely to "see the place," but Gupta, a man of action and generosity, took him to meet the Principal. "You start teaching tomorrow," the Principal declared. Unprepared, with few belongings and no place to stay, Bambah was hesitant. But Gupta, with a kindness that would define their lifelong friendship, offered him a bed in his own home and even lent him shirts. This three-month period in Hoshiarpur solidified a deep bond between Bambah and the Gupta family.

In 1948, while teaching at Delhi University, he was awarded the prestigious Royal 1851 Exhibition Fellowship, which took him to the University of Cambridge. There, he completed his Ph.D. in just two years - an extraordinary academic achievement. Returning to India in 1951, he joined Delhi University before moving to Panjab University (Hoshiarpur campus) in 1952 as a Reader. Around the same time, he also held a fellowship at the Institute for Advanced Study, Princeton, where he spent two enriching years before returning to Panjab University.

At Panjab University, where he spent most of his career, he didn't just lead; he nurtured. His extraordinary partnership with Professor Hansraj Gupta was a rare psychological feat in academia, a field often marred by ego and competition. Their ability to work in harmony, transcending differences in seniority, speaks to a shared growth mindset and a secure sense of identity. They were not threatened by each other's success but saw it as a collective win, a powerful example of collaborative leadership that laid the foundation for a vibrant mathematical tradition.

Together with Professor Hans Raj Gupta, he laid the foundation of what came to be known as the Punjab School of Mathematics, continuing the legacy of Prof. Chowla. After further research engagements in the U.S., including at the University of Notre Dame and Ohio State University, Prof. Bambah returned to Panjab University, where he was promoted to Professor in 1958. Under his leadership, the Department of Mathematics became the first Centre for Advanced Study in 1963 in the Indian university system, and this status continued until three years ago, when the UGC discontinued the scheme for all universities.

Bambah's work spanned several areas: number theory, discrete geometry, and lattice problems. He earned a D.Sc. from Cambridge in 1970, was elected an Emeritus Professor of Panjab University in 1993, and was awarded a D.Sc. (Honoris Causa) by PU in 2016. Among the many national honours bestowed upon him were the Padma Bhushan and the Srinivasa Ramanujan Medal, acknowledging his lifelong service to mathematics and higher education.

He also served as President of the Indian Mathematical Society and as Vice-Chancellor of Panjab University (1985-1991). His VC tenure coincided with political unrest in Punjab and his approach embodied high emotional intelligence. He steered the institution through a phase of significant academic and infrastructural advancement. His leadership, vision, and unwavering commitment left an enduring legacy-not only on the university but also on the generations of students and colleagues he inspired and mentored.

He believed in encouraging the young. "Younger people are naturally better," he would say. "They stand on your shoulders, and therefore get a more commanding view. Instead of being jealous, you should encourage them." This philosophy, combined with his deep sense of institutional responsibility, defined his career.

As an academic leader, he often spoke candidly about creativity and education. He lamented the stifling effect of bureaucratic controls on universities, arguing that true creativity required freedom. He admired the University Grants Commission of earlier decades, when it once granted him eight new faculty positions without restrictions.

Even after retirement, Prof. Bambah remained intellectually vibrant, deeply curious, and actively engaged with new mathematical developments. What set him apart was not just his towering intellect but also his humility, kindness, and unyielding dedication to public service. His final act of generosity - the donation of his body for medical research at PGIMER - epitomizes the values he lived by.

As we reflect on his life, let us remember Professor R. P. Bambah not only for his unmatched scholarly achievements but also for his grace, generosity, and vision. His legacy will continue to illuminate the path for generations of mathematicians and educators. His passing leaves a void, but his legacy lives on in the institutions he nurtured, the students he inspired, and the profound impact he had on the landscape of mathematics.

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An extract from: "R. P. Bambah in Conversation with C. S. Aravinda and Sudhir Rao" in Bhavana volume 4, Issue 1, January, 2020.

You also received the Padma Bhushan way back in 1988 itself.

RPB: Yes. The Padma Bhushan was again an accident because the Ramanujan centenary was going on at the time. At Madras, there was this meeting when MGR [M.G. Ramachandran, former Chief Minister of Tamil Nadu] had just then passed away. The main function was held there, and Prime Minister Rajiv Gandhi and Education Minister P.V. Narasimha Rao were there too. My friends from TIFR decided to put me up on the dais. They could have put up other people, too. Anyway, I was one of the persons on the dais and Rajiv Gandhi must have thought that since a mathematician's centenary was being celebrated, the honour [Padma Bhushan] should go to mathematics. I have no idea what else happened.

Courtesy *Bhāvanā* and Prof. Bindu Bambah

9. “Mathematics and Society” as Envisioned by Professor R. P. Bambah

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The reader would have learnt from various articles in this issue about the rich legacy left by Professor Bambah to the mathematical community, in terms of his achievements as a mathematician, an educator, as well as an administrator. Many meritorious features of his personality, have been commented upon in the articles, by knowledgeable scholars who also had the benefit of being well-acquainted with him in person, in various roles. The aim of this brief article is to bring out his vision on the theme of “Mathematics and Society”; the title is borrowed from his address [3], which is one of the main planks of this write-up.

Through his illustrious career as a leading mathematician Prof. Bambah had various special occasions to address the wider mathematical community, and put forward his thoughts on the progress of mathematics and the role of a mathematician, with a review and celebration of major achievements in mathematics, and formulation of tasks for the future. The endeavour involved also relating the activity to the society as a whole. It is an enriching experience to go over his articles [1], [2] and [3] in this respect, corresponding to his addresses. [1]

In the preamble to [2], his Presidential Address at the 48th Annual Session at Guwahati in 1978, we get an insightful glimpse into his perspective on the role of a mathematician in the society. In his words:

> ... There is a realisation of the fact that although extension and expansion of knowledge is essential, one can not leave to chance the impact of these on Society; there has to be a planned interaction between the universities and Society for which the growth of knowledge has to act as an active agent of progress and social change. There is also an implicit assumption that the Scientific and Academic community should take up problems of immediate relevance to the development of economic, moral and social progress of the country. As a citizen and educator I also subscribe to the above point of view. However, I would like to add that, while giving priority to problems of relevance and applicability, we should not forget that historically great scientific progress has often been made as a response to challenges of hard unsolved problems, to the desire for search for aesthetically satisfying simple patterns to explain complicated phenomena and, most of all, out of a sense of curiosity to know and explore. ...

In the article he goes on to explain, albeit briefly, on some old problems settled around that time, whose solutions he mentions to have “given the mathematical community a great sense of satisfaction and achievement”. After a discussion on some of the major problems in various branches of mathematics (some of these are mentioned below in the context of [3], so I am skipping the details here), the article concludes with an introduction to the topic of covering and packing problems in Geometry of numbers, which he specialized in and made some notable contributions.

A detailed profile of the historical development of mathematics, and how it has been helpful in shaping various sciences is found in his address [3] in a seminar organized by INSA in 1978. An interesting and unique feature of the address is the author’s statement “ ... I accepted the President’s invitation with the understanding conveyed to him that I shall prepare this talk with the help of various other colleagues in the country. I wrote to some of my friends and in their generosity many of them sent very useful comments to me. The address, therefore, can be looked upon as a collective effort ... ”; he mentions eight names in this respect (which I need not repeat here).

[1] I would also recommend to the reader his article [4], discussing various issues concerning the quality of science in India, an outcome of a project in a similar spirit, which manifests similar commitment and thoroughness in its execution. The article is however beyond the scope of the present theme and will not be discussed here. Also, [1] is interesting in its own right, but its contents are, in a way, largely subsumed in the later papers, so I will not be referring to it separately, and will focus on [2] and [3].

This illustrates his open-mindedness to ideas, scholarly attitude, and a high level of professional integrity, as a leader. There is no doubt however that the outcome bears a strong stamp of his perspective and style of communication.

The address has 9 sections, and it would be instructive to list them here: 1. Introduction; 2. Mathematics and Physics, Chemistry; 3. Mathematics and Economics, Planning, Industry, Management; 4. Mathematics and Environment; 5. Mathematics and Biology; 6. Mathematics and Communication; 7. Role of Statistics and Probability; 8. Computers and Society; 9. Mathematics and Society in India; the exposition is accompanied by an informative table and several references.

The Introduction puts the subject in perspective. The author mentions in particular,

> I would like to point out that every success in the subject is a tribute to the human spirit. When a problem which had resisted the attempts by many great minds in the past over decades or even centuries gets solved, the whole human race shares the glory. ...

He recalls in this spirit several developments including the solution of the four-colour problem (Appel and Haken), the Ramanujan conjecture (Deligne), Chen's work on the Goldbach conjecture, Hilberts 10th problem on Diophantine equations (Matijasevic), Independence of the axiom of choice and the continuum hypothesis (Paul Cohn), and concludes with the comment “These are some of the great events which fill one with pride in the powers of the human intellect.”

At the start of § 3 on Mathematics and Economics, Planning, Industry, Management, it is pointed out how the axiomatic method developed by mathematicians is used fruitfully by natural scientists and how “The clear understanding of this process in the early part of the [20th] century had its effect on other branches of human interest, like economics, planning, industry, management etc., in the 1930's and more extensively during and after the second World War. Numerous names involved in the process are mentioned.

The remaining sections are relatively brief, but incisive, and would serve as good reference points in any discussion on the topic. The role played in the subsequent developments by the availability of high speed computers is highlighted. In the section relating to Biology, Zeeman's work on Catastrophe theory finds special mention. In the final section on Mathematics and Society, in India, after mentioning work of various groups in the country, the author includes the following reminder:

> ... mathematics transcends national boundaries and international programmes for the young may create bonds and relationships between highly gifted young people of various countries, which in the future may, in the words of a senior educationist, prevent international disasters, because an international community of highly gifted people with complete understanding will have come into existence.

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An International Conference on the theme ``**Algebra and Number Theory - Celebrating 100 years of Professor R. P. Bambah**" was held at the Department of Mathematics, Panjab University Chandigarh, during September 30 - October 1, 2025.

The Conference was inaugurated by Prof. Yojana Rawat, Dean University Instruction, Panjab University. There were talks by several speakers from across the country and also members from the School of Geometry of Numbers associated with Professor Bambah. A report on the conference, by Prof. Surinder Pal Singh Kainth, was published in the October, 2025 issue of TMCB.

At the inauguration of the conference (from left to right): Professors A. K. Agarwal, S. P. S. Kainth, S. K. Tomar, R. J. Hans Gill, Yojana Rawat, Madhu Raka, and M. S. Raghunathan.

Mathematicians who had significant impact on Prof. Bambah's Research

Hermann Minkowski (12 June 1864 - 12 Jan. 1909)

A German mathematician, the founder of the Geometry of Numbers, a main area of research of Prof. Bambah. Minkowski formalized it as a distinct branch of mathematics. His primary insight was that problems in number theory could be solved by viewing them through the lens of n-dimensional geometry. His work revolutionized several areas of mathematics like Diophantine Approximation, Algebraic Number theory, the General theory of relativity.

Louis Joel Modrell (28 Jan. 1888 - 12 March 1972)

A titan of 20th-century number theory, a Ph.D. supervisor of Prof. Bambah. He significantly advanced the Geometry of Numbers. He transformed it into a powerful tool for solving complex Diophantine equations. In 1922, Mordell proved that the group of rational points on an elliptic curve is finitely generated. He specifically worked on the "Object of the Geometry of Numbers". He wrote over 270 papers and the classic textbook Diophantine Equations (1969).

Harold Davenport (30 Oct. 1907 - 09 June 1969)

An English mathematician who is much-cited in Prof. Bambah's thesis. He worked on the geometry of numbers, Diophantine approximation and the analytic theory of numbers. He wrote a number of important textbooks and monographs including The Higher Arithmetic (1952). He refined precision of the Geometry of Numbers and applied it to the most challenging problems.

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